conclude that $c_n = 0$ for $n > 0$. Hence f has the expansion

$$f(z) = \sum_{n=-\infty}^{0} c_n z^n \qquad (|z| > R).$$

Singularities

6.5 Definitions

Let f be a complex-valued function. The point a is a *regular point* if f is holomorphic at a (that is, if there exists r such that $f \in \mathrm{H}(\mathrm{D}(a\,;r))$; see 2.2(3)). The point a is a *singularity* of f if a is a limit point of regular points which is not itself regular.

If a is a singularity of f and f is holomorphic in some punctured disc $\mathrm{D}'(a; r)$, then a is an *isolated singularity*; if $f \notin \mathrm{H}(\mathrm{D}'(a\,;r))$ for any $r > 0$, a is a *non-isolated essential singularity*.

6.6 Classification of isolated singularities

Suppose f has an isolated singularity at a. Then f is holomorphic in some annulus $\{z : 0 < |z-a| < r\}$ and there has a *unique* Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

The point a is said to be:

a removable singularity if $c_n = 0$ for all $n < 0$;
a *pole of order m* ($m \geq 1$) if $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$;
an *isolated essential singularity* if there does not exist m such that $c_n = 0$ for all $n < -m$.

Poles of orders 1, 2, 3, ... are called *simple, double, triple, ...*

Notes (1) Uniqueness of the Laurent coefficients ensures that these definitions make sense.
(2) In $\mathrm{D}'(a; r)$,

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z-a)^n + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

The first sum on the right-hand side is the *principal part* of the Laurent expansion; the second sum is holomorphic in $\mathrm{D}(a\,;r)$, by 2.12. Notice that $f(z) - \sum_{n=-\infty}^{-1} c_n (z-a)^n$ has a removable singularity at a. For more information on removable singularities see 6.12(1).

6.7 Examples

(1) $(z-1)^{-2}$ has a double pole at $z = 1$.
(2) $(1-\cos z)z^{-2}$ is holomorphic except at $z = 0$, where it is inde-

terminate. The Laurent expansion about $z = 0$ is

$$\frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots,$$

so the singularity at 0 is removable.
(3) We showed in Example 6.4(4) that

$$\cot z = \frac{1}{z} - \frac{z}{3} + O(z^3) \qquad (z \in \mathrm{D}'(0; \pi)).$$

Hence $\cot z$ has a simple pole at 0. Since $\cot(z - k\pi) = \cot z$ for each integer k, each singularity $k\pi$ of $\cot z$ is a simple pole.
(4) If $0 < |z| < \infty$,

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{-(2n+1)}}{(2n+1)!}.$$

Hence $\sin(1/z)$ has an isolated essential singularity at 0.
(5) $\operatorname{cosec}(1/z)$ has singularities at $1/(k\pi)$ ($k \in \mathbb{Z}$). For $k \neq 0$ there is a simple pole at $1/(k\pi)$. Since $\operatorname{cosec}(1/z)$ is not holomorphic in any punctured disc $\mathrm{D}'(0; r)$, the point 0 is *not* an isolated singularity. See also 6.15.

It should be clear from the examples in 6.4 that direct computation of the Laurent coefficients is an arduous way of classifying the singularities of even relatively simple functions. The clue to a more efficient method lies in the observation that, if a holomorphic function has an isolated zero at the point a, then its reciprocal has an isolated singularity at a. To exploit this to the full we need some preliminary facts about zeros, and a technical theorem.

6.8 Zeros

Suppose that $f \in \mathrm{H}(\mathrm{D}(a\,;r))$ for some r and that $f(a) = 0$. Assume that f is not identically zero in $\mathrm{D}(a; r)$. By Taylor's theorem, 5.9,

$$f(z) = \sum_{n=m}^{\infty} c_n (z-a)^n \qquad (z \in \mathrm{D}(a; r)), \quad \text{where} \quad m \geq 1 \text{ and } c_m \neq 0.$$

We define the *order* of the zero of f at a to be m. Zeros of orders 1, 2, ... are called *simple, double, ...* Since, by 2.13, $c_n = f^{(n)}(a)/n!$, f has a zero of order m at a if and only if

$$f(a) = f'(a) = \dots = f^{(m-1)}(a) = 0, \qquad f^{(m)}(a) \neq 0.$$

6.9 Theorem

(1) Let $f \in \mathrm{H}(\mathrm{D}(a\,;r))$. Then f has a zero of order m at a if and only if

$$\lim (z-a)^{-m} f(z) = C \ldots$$

(2) Let $f \in H(D'(a;r))$. Then f has a pole of order m at a if and only if

$$\lim_{z \to a}(z-a)^m f(z) = D, \text{ where } D \text{ is a non-zero constant.} \qquad (\dagger)$$

Proof. We prove (2). The proof of (1) is very similar, and is left as an exercise.

Necessity Suppose a is a pole of order m. For $z \in D'(a;r)$,

$$f(z) = \sum_{n=-m}^{\infty} c_n(z-a)^n, \quad \text{where} \quad c_{-m} \neq 0.$$

In $D'(a;r)$, $(z-a)^m f(z) = \sum_{n=0}^{\infty} c_{n-m}(z-a)^n$. The series on the right-hand side defines a function continuous at $z=a$ (by 2.12 and 2.3(1)). Therefore

$$\lim_{z \to a}(z-a)^m f(z) = c_{-m} \neq 0.$$

Sufficiency By Laurent's theorem, 6.1,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n, \quad \text{where}$$

$$c_n = \frac{1}{2\pi i}\int_{\gamma(a;s)} \frac{f(w)}{(w-a)^{n+1}}\,dw \quad (0<s<r).$$

We require $c_n = 0$ $(n<-m)$, $c_{-m} \neq 0$. By the condition (†), given $\varepsilon>0$, there exists $\delta>0$ such that

$$|(w-a)^m f(w) - D| < \varepsilon \quad \text{whenever} \quad 0<|w-a|<\delta.$$

Take $0<s<\min\{\delta, r\}$. Then

$$|w-a| = s \Rightarrow |(w-a)^m f(w)| \leq |D| + \varepsilon \quad \text{(by 1.4(2))}$$
$$\Rightarrow |(w-a)^{-n-1} f(w)| \leq (|D|+\varepsilon)s^{-n-m-1}.$$

Hence, by estimating the integral defining c_n (using 3.10)),

$$|c_n| \leq (|D|+\varepsilon)s^{-n-m}.$$

If $n<-m$, s^{-n-m} can be made arbitrarily small by taking s sufficiently small. The constant c_n is independent of s, so $c_n = 0$.

We now have $f(z) = \sum_{n=-m}^{\infty} c_n(z-a)^n$. As in the necessity proof,

$$c_{-m} = \lim_{z \to a}(z-a)^m f(z) = D \neq 0.$$ □

6.10 Corollaries

(1) Suppose f is holomorphic in some disc $D(a;r)$. Then f has a zero of order m at a if and only if $1/f$ has a pole of order m at a.

(2) Suppose that f has a pole of order m at a.

(i) Suppose $g \in H(D(a;r))$ for some r. Then, at a, the function fg has

a pole of order m if $g(a) \neq 0$,

a pole of order $m-n$ if g has a zero of order n at a and $n<m$,

a removable singularity if g has a zero of order at least m.

(ii) Suppose g has a pole of order n at a. Then fg has a pole of order $m+n$ at a.

Proof. (1) Suppose $1/f$ has a pole at a. Then f cannot have a non-isolated zero at a, and so, by the Identity theorem, 5.14, f is non-zero in $D'(a;s)$ for some $s>0$. The result (sufficiency and necessity) now follows from Theorem 6.9.

The proof of (2) is left as an exercise. □

6.11 Examples

(1) $z \sin z$ has zeros at $z = n\pi$ $(n \in \mathbb{Z})$.

$$\left[\frac{d}{dz}(z\sin z)\right]_{z=0} = \left[(\sin z + z\cos z)\right]_{z=0} = 0, \left[\frac{d^2}{dz^2}(z \sin z)\right]_{z=0} \neq 0,$$

and for $n \neq 0$,

$$\left[\frac{d}{dz}(z\sin z)\right]_{z=n\pi} \neq 0.$$

Hence, by 6.8 and 6.10, $1/(z \sin z)$ has a double pole at 0 and simple poles at $z = n\pi$ $(n \in \mathbb{Z},\ n \neq 0)$.

(2) Consider cot z. At the points $n\pi$ $(n \in \mathbb{Z})$, $\sin z$ has simple zeros (by 6.8) and $\cos z \neq 0$. Hence, by 6.10, $\cot z$ has simple poles at $z = n\pi$ $(n \in \mathbb{Z})$. Compare this method with that of 6.7(3).

(3) Consider

$$F(z) = \frac{(z-1)^2 \cos \pi z}{(2z-1)(z^2+1)^5 \sin^3 \pi z}.$$

The denominator has a simple zero at 1/2, zeros of order 5 at $\pm i$ and a triple zero at k for each $k \in \mathbb{Z}$; the numerator has a double zero at 1 and simple zero at $(2k+1)/2$ for each $k \in \mathbb{Z}$. Appealing to 6.10 we see that f has poles of order 5 at $\pm i$, a simple pole at 1, a triple pole at k, for $k \in \mathbb{Z}$, $k \neq 1$, and a removable singularity at 1/2.

6.12 Behaviour near an isolated singularity

(1) **Removable singularity** Suppose $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ in $D'(a;r)$. Then $f(z) \to c_0$ as $z \to a$. By defining (or redefining) $f(a)$

to be c_0, we make

$$f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n \quad \text{in } \mathrm{D}(a;r),$$

and so make f holomorphic in $\mathrm{D}(a;r)$, by 2.12. Thus a removable singularity is something of a non-event: a ceases to be classified as a singularity once f is correctly defined at a.

(2) **Pole** Suppose f has a pole at a. It is immediate from Corollary 6.10(1) that $|f(z)|\to\infty$ as $z\to a$.

(3) **Essential singularity** Suppose f has an isolated essential singularity at a. Let w be *any* complex number. Then there exists a sequence $\langle a_n\rangle$ such that $a_n\to a$ and $f(a_n)\to w$. This is the *Casorati–Weierstrass theorem*. For an outline of the proof, see Exercise 6.10.

A more spectacular and much deeper result, due to Picard, asserts that in any $\mathrm{D}'(a;r)$, f actually assumes every complex value, except possibly one. In the case of $e^{1/z}$, which has an isolated essential singularity at 0, the exception is 0.

Meromorphic functions

6.13 The extended complex plane

For the analysis of the behaviour of holomorphic functions as $|z|\to\infty$ we introduce a convenient device, the *extended complex plane* $\tilde{\mathbb{C}}$. We form $\tilde{\mathbb{C}}$ by adding an extra point ∞ to $\mathbb{C}$. We can give our construction geometrical significance as follows: we regard $\mathbb{C}$ as embedded in Euclidean space $\mathbb{R}^3$ by identifying $z=x+iy$ with $(x, y, 0)$. We let

$$\Sigma:=\{(x,y,u)\in\mathbb{R}^3 : x^2+y^2+(u-\tfrac{1}{2})^2=\tfrac{1}{4}\};$$

this is a sphere (the *Riemann sphere*), touching the plane $\mathbb{C}$ at $(0,0,0)$. Stereographic projection (see Fig. 6.2) allows us to set up a one-to-one correspondence between $\tilde{\mathbb{C}}$ and Σ, under which

$$\mathbb{C}\ni z=x+iy=re^{i\theta}\leftrightarrow z=(x(1+r^2)^{-1}, y(1+r^2)^{-1}, r^2(1+r^2)^{-1}),$$

$$\infty\leftrightarrow(0,0,1), \text{ the north pole of } \Sigma.$$

We define open discs in $\tilde{\mathbb{C}}$ as follows:

$$\mathrm{D}(a;r):=\{z\in\mathbb{C} : |z-a|<r\} \quad (a\in\mathbb{C}, r>0),$$

$$\mathrm{D}(\infty;r):=\{z\in\mathbb{C} : |z|>r\}\cup\{\infty\} \quad (r>0).$$

A subset G of $\tilde{\mathbb{C}}$ is said to be *open* if, given $z\in G$, there exists r

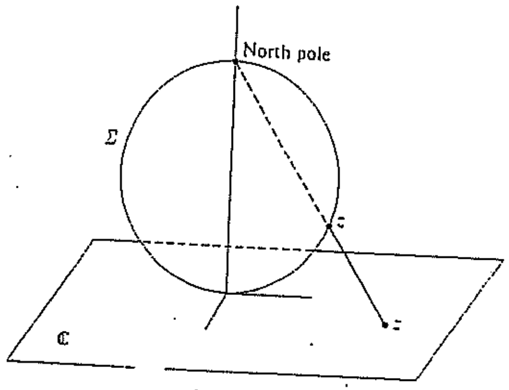

Fig. 6.2

such that $\mathrm{D}(z;r)\subseteq G$. [The open sets in $\tilde{\mathbb{C}}$ correspond to the intersections with Σ of open spheres in $\mathbb{R}^3$. We have defined a topology on $\tilde{\mathbb{C}}$ so as to make $\tilde{\mathbb{C}}$ homeomorphic to the sphere Σ. As a closed bounded subset of $\mathbb{R}^3$, Σ is compact, so $\tilde{\mathbb{C}}$ is compact, and can be regarded as a one-point compactification of the non-compact space $\mathbb{C}$.]

Continuity, etc., of functions on $\tilde{\mathbb{C}}$ can be handled topologically. More naively, we may decree that a complex-valued function f defined on a set containing some disc $\mathrm{D}(\infty;r)$ is *continuous* (*differentiable*) at ∞ if and only if $\tilde{f}$ defined on $\mathrm{D}(0;1/r)\subseteq\mathbb{C}$ by

$$\tilde{f}(z)=f(1/z)\ (z\neq 0), \qquad \tilde{f}(0)=f(\infty)$$

is continuous (differentiable) at 0. Similarly, all the terms relating to zeros and singularities are applied to ∞. For example, f is said to have a *pole at* ∞ if and only if $\tilde{f}$ has a pole at 0; e.g. at ∞, z^3 has a triple pole and $1/z^2\sin(1/z)$ a removable singularity.

6.14 Definition

Let $G\subseteq\tilde{\mathbb{C}}$ be open. A complex-valued function which is holomorphic in G except possibly for poles is said to be *meromorphic* in G.

6.15 Limit points of singularities

Closed sets and limit points in $\tilde{\mathbb{C}}$ are defined in the same way as in $\mathbb{C}$; see 1.10. We claim that any infinite closed subset S of $\tilde{\mathbb{C}}$ has a limit point in S. We prove this from the Bolzano–Weierstrass theorem as presented in 1.17. If there exists R such that $S\subseteq\bar{\mathrm{D}}(0;R)$, our claim follows immediately from 1.17. Otherwise, $S\cap\{z\in\mathbb{C} : |z|>r\}\neq\varnothing$ ...