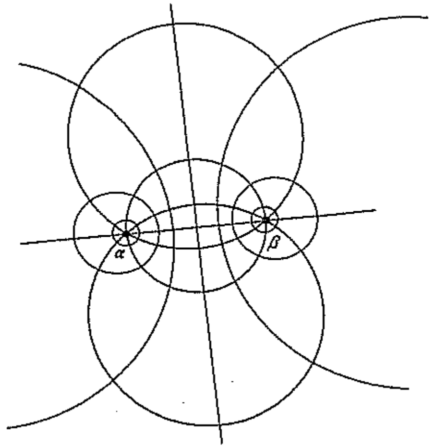


Fig. 10.3

Conformal mapping

We now investigate how holomorphic functions behave when considered as geometric mappings.

10.6 Theorem

Suppose f is holomorphic in an open set G, $\zeta \in G$, and $f'(\zeta) \neq 0$. Then, in the sense defined below, f preserves angles between paths in G meeting at ζ.

Proof. Let γ_1 and γ_2 be paths lying in G, both with parameter interval $[0, 1]$, having common endpoint $\zeta = \gamma_1(0) = \gamma_2(0)$. We suppose that, for $k = 1$ and 2, $\gamma_k'(0) \neq 0$, so that γ_k has a well-defined tangent at ζ given by $\gamma_k(t) = \zeta + \gamma_k'(0)t$ $(t \geqslant 0)$ and making an angle $\arg \gamma_k'(0)$ with the real axis. The angle between γ_1 and γ_2 is then (by definition) $\lambda = \arg \gamma_1'(0) - \arg \gamma_2'(0)$.

The paths γ_1 and γ_2 are mapped by f to paths $f \circ \gamma_1$ and $f \circ \gamma_2$ and these meet at $f(\zeta)$ at an angle $\Lambda = \arg(f \circ \gamma_1)'(0) - \arg(f \circ \gamma_2)'(0)$. The assertion of the theorem is that $\Lambda = \lambda \pmod{2\pi}$. By the chain rule,

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{f'(\zeta)\gamma_1'(0)}{f'(\zeta)\gamma_2'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)},$$

from which the result follows; see 2.18. □

10.7 Definition

A mapping f is *conformal in an open set* G if $f \in H(G)$ and $f'(z) \neq 0$ for any $z \in G$; f is *conformal at a point* ζ if it is conformal in some disc $D(\zeta; r)$.

The proof of Theorem 10.6 shows that conformal mappings preserve both the magnitude and sense of angles. If f is differentiable at ζ but $f'(\zeta) = 0$, f does not preserve angles at ζ. Consider, for example, $f(z) = z^2$, which doubles angles at 0.

10.8 Construction of conformal mappings: preliminary remarks

(1) Suppose we require a conformal map f from the open upper half-plane $\Pi = \{z : \operatorname{Im} z > 0\}$ onto the open unit disc $D(0; 1)$. It is unlikely to be helpful to bring $\operatorname{Im} z$ (a non-holomorphic function) directly into the definition of f (which must be holomorphic). Recall however that $\Pi = \{z : |z - i| < |z + i|\}$ (the set of points closer to i than to $-$i). It ought now to be clear that we should take $f(z) = (z - i)/(z + i)$ for then

$$z \in \Pi \Leftrightarrow |f(z)| < 1 \Leftrightarrow f(z) \in D(0; 1).$$

Also f is conformal in Π since f is holomorphic there, with $f'(z) = 2i(z + i)^{-2} \neq 0$. This simple example shows that success in constructing a conformal mapping from one region to another may depend on a judicious choice of descriptions for the regions. Later examples will reinforce this point.

(2) The composition of conformal mappings is conformal, by the chain rule. Hence we can hope to build up a conformal mapping from one region to another by taking a finite sequence of 'elementary' conformal mappings. For example, a typical sequence of mappings from a lozenge (bounded by circular arcs) to $D(0; 1)$ might be as shown in Fig. 10.4. Thus an aid to successful map-building is familiarity with standard mappings. These include the

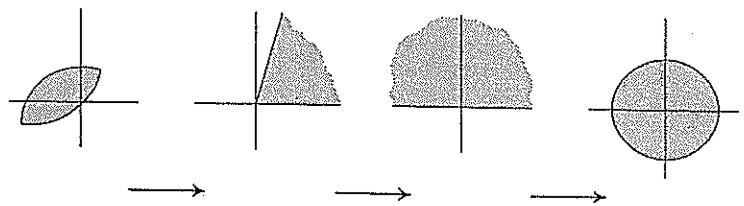


Fig. 10.4

Möbius transformations, exponentials, and powers discussed below.

Möbius transformations

10.9 Definition

A Möbius transformation is a mapping of the form

$$f: z \mapsto \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{C},\ ad-bc \neq 0).$$

(The excluded case $ad-bc=0$ produces a constant mapping.)

The Möbius transformation f is best viewed as a mapping of $\tilde{\mathbb{C}}$ to itself, with (by definition) $f(-d/c)=\infty$, $f(\infty)=a/c$. The map $f:\tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ is one-to-one and onto, with inverse

$$f^{-1}: w \mapsto \frac{dw-b}{a-cw}$$

also a Möbius transformation. It is easily checked that the Möbius transformations form a group under the operation of composition of maps. A general Möbius transformation can be built up from mappings of the following types:

$z \mapsto ze^{i\phi}$ (ϕ real)	(anticlockwise rotation through ϕ),
$z \mapsto Rz$ ($R>0$)	(stretching by a factor of R),
$z \mapsto z+a$ ($a \in \mathbb{C}$)	(translation by a),
$z \mapsto 1/z$	(inversion).

Suppose $f(z)=(az+b)/(cz+d)$. Then $f'(z)=(ad-bc)/(cz+d)^2$, which shows that f is conformal on $\mathbb{C}\setminus\{-d/c\}$. Theorem 10.10 shows just how much room for manoeuvre we have in constructing Möbius transformations.

10.10 Theorem

Suppose each of $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ is a triple of distinct points in $\tilde{\mathbb{C}}$. Then there exists a unique Möbius transformation f such that $f(z_k)=w_k$ $(k=1,2,3)$, given by $f: z \mapsto w=f(z)$, where

$$\left(\frac{w-w_1}{w-w_3}\right)\left(\frac{w_2-w_3}{w_2-w_1}\right)=\left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right).$$

Proof. The map

$$g: z \mapsto \left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right)$$

takes z_1, z_2, and z_3 to $0, 1$, and ∞, respectively. Construct h in the same way as g, to map w_1, w_2, and w_3 to $0, 1$, and ∞. Then $f=h^{-1}\circ g$ is the map in the statement of the theorem and $f(z_k)=w_k\,(k=1,2,3)$.

For uniqueness it is enough to show that the only Möbius transformation $f: z \mapsto (az+b)/(cz+d)$ fixing 0, 1, and ∞ is the identity map. The conditions $f(0)=0$, $f(\infty)=\infty$, and $f(1)=1$ force in turn $b=0$, $c=0$, and $a=d$, so that $f(z)=z$ for all z. □

One can show that Möbius transformations map circlines to circlines by using any of the standard representations of circles and lines. The proof below shows more: the preservation of inverse points.

10.11 Theorem

Let S be a circline with inverse points α and β $(\alpha, \beta \in \mathbb{C}, \alpha \neq \beta)$ and let f be a Möbius transformation. Then f maps S to a circline with inverse points $f(\alpha)$ and $f(\beta)$.

Proof. We use Theorem 10.2 to write the equation of S in the form $|(z-\alpha)/(z-\beta)|=\lambda$. Suppose $w=f(z)=(az+b)/(cz+d)$, so $z=(dw-b)/(a-cw)$. Then

$$\left|\frac{(\alpha c+d)w-(\alpha a+b)}{(\beta c+d)w-(\beta a+b)}\right|=\lambda.$$

We may rewrite this as

(i) $\left|\dfrac{w-f(\alpha)}{w-f(\beta)}\right|=\lambda\left|\dfrac{\beta c+d}{\alpha c+d}\right|$, if $\alpha c+d\neq 0$ and $\beta c+d \neq 0$, or

(ii) $|w-f(\alpha)|=\lambda\left|\dfrac{\beta a+b}{\alpha c+d}\right|$, if $\alpha c+d\neq 0$ and $\beta c+d=0$, or

(iii) $|w-f(\beta)|=\dfrac{1}{\lambda}\left|\dfrac{\alpha a+b}{\beta c+d}\right|$, if $\alpha c+d=0$ and $\beta c+d\neq 0$.

(Note that $\alpha c+d$ and $\beta c+d$ cannot both be zero.) In each case the image of S is a circline with $f(\alpha)$ and $f(\beta)$ as inverse points; in cases (ii) and (iii) the images of the original inverse points are the centre of the circle and the point ∞. □

10.12 Theorem

There exists a Möbius transformation mapping any given circline to any other given circline.

Proof. The result follows immediately from Theorems 10.10 and 10.11, since a circline is uniquely determined by three points (these may be taken to be a pair of inverse points and one point on the circline, or three points on it). □

10.13 Examples

(1) To find the images under $f: z \mapsto w = (z-1)^{-1}$ of (a) the real axis, (b) the imaginary axis, (c) the circle centre 0, radius r.

Solution (a) The real axis is mapped to the circline through $f(0) = -1$, $f(1) = \infty$, $f(\infty) = 0$; viz. the real axis. (Alternatively one may use the fact that the real axis has equation $z = \bar{z}$.)
(b) The imaginary axis has equation $|z-1| = |z+1|$ and is mapped to the circline with equation $|2w+1| = 1$, which is the circle with centre $-\frac{1}{2}$ and radius $\frac{1}{2}$.
(c) The required image has equation $|w+1| = r\,|w|$. If $r = 1$, it is the line $\operatorname{Re} w = -\frac{1}{2}$. If $r \neq 1$, it is a circle with -1 and 0 as inverse points. It has the points $(\pm r - 1)^{-1}$ as the ends of a diameter (see Fig. 10.1), and hence has centre $(r^2-1)^{-1}$ and radius $r\,|1-r^2|^{-1}$. □
(2) To find all Möbius transformations mapping the unit circle T to itself and mapping α to 0.

Solution. Let f satisfy the required conditions and map 1 to $e^{i\phi} \in$ T. The points α and $1/\bar{\alpha}$ are inverse points with respect to T and so are mapped by f to 0 and ∞ (by 10.11). The unique Möbius transformation taking α, 1, and $0/\bar{\alpha}$ to 1, $e^{i\phi}$, and ∞, respectively, is given by Theorem 10.10 to be $z \mapsto w$ where

$$\frac{w}{e^{i\phi}} = \left(\frac{z-\alpha}{z-1/\bar{\alpha}}\right)\left(\frac{1-1/\bar{\alpha}}{1-\alpha}\right) = \left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)\left(\frac{\bar{\alpha}-1}{1-\alpha}\right).$$

But $|1-\alpha| = |1-\bar{\alpha}|$, so, for some real number ψ,

$$w = e^{i\psi}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right).$$

□

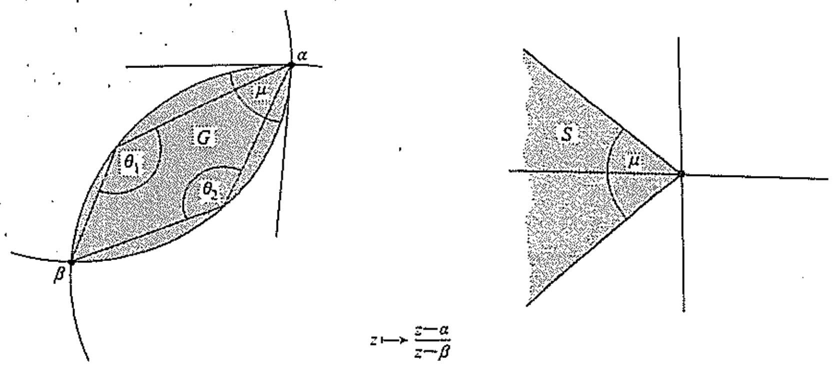


Fig. 10.5

10.14 Mappings of regions bounded by circular arcs

A Möbius transformation can be used, *inter alia*, to map a lozenge-shaped region bounded by circular arcs onto a sector. Let G be as shown in Fig. 10.5. By 10.4,

$$G = \left\{z : \theta_1 < \arg\left(\frac{z-\alpha}{z-\beta}\right) < 2\pi - \theta_2\right\}.$$

The Möbius transformation $f: z \mapsto (z-\alpha)/(z-\beta)$ maps G conformally onto the sector

$$S = \{w : \theta_1 < \arg w < 2\pi - \theta_2\}.$$

Note the effect of mapping β to ∞; circular arcs ending at β transform to half-lines. Observe also that, because f is conformal at α, the angle subtended by S at 0 equals the angle μ between the bounding arcs of G. Note that the image sector is determined completely by the image of a single point other than α or β. By taking the map $z \mapsto k(z-\alpha)/(z-\beta)$, with $k = e^{i\varphi}$ ($\varphi \in \mathbb{R}$) chosen suitably, the image sector can be swung round into any desired position.

This argument deals with mappings of regions bounded by two members of a coaxal system $C_2(\alpha, \beta)$ as defined in 10.5. Exercise 10.8 concerns the mapping of a region bounded by two non-concentric circles, which form members of some coaxal system $C_1(\alpha, \beta)$.

Other mappings: powers, exponentials, and the Joukowski transformation

10.15 Powers

The map $z \mapsto z^n$ $(n=2,3,\ldots)$ is conformal except at 0, where angles between paths are magnified by a factor of n. With non-integer powers we have to contend with a multifunction. Suppose, for definiteness, that we define, for $\alpha \in \mathbb{R}$, $z^\alpha = |z|^\alpha e^{i\theta\alpha}$ $(z=|z|e^{i\theta}, -\pi<\theta\leqslant\pi)$. Then $z\mapsto z^\alpha$ is conformal in the plane cut along $(-\infty, 0]$. Particularly useful is $z\mapsto z^{\pi/\beta}$; for $0<\beta<\pi$, this takes the sector $\{z: 0<\arg z<\beta\}$ conformally onto the open upper half-plane (see Fig. 10.6).

10.16 Exponentials

Let $f: z = x+iy \mapsto e^z = w = Re^{i\phi}$; this map is conformal in $\mathbb{C}$. Since $R=e^x$ and $\phi = g \pmod{2\pi}$, f maps

a line $x=a$ to a circle $|w|=e^a$, and

a line $y=c$ to a half-line $\arg w = c$.

Hence f takes the vertical strip $\{z: a<\operatorname{Re} z<b\}$ to the annulus $\{w: e^a<|w|<e^b\}$ and the horizontal strip $\{z: c<\operatorname{Im} z<d\}$ to the sector $\{w: c<\arg w<d\}$; see Fig. 10.7. In reverse, a logarithm will map sectors to strips, but we need to select a holomorphic branch and to work in the appropriate cut plane.

10.17 The Joukowski transformation

Möbius transformations, powers and exponentials have the property of mapping certain families of circles and straight lines to

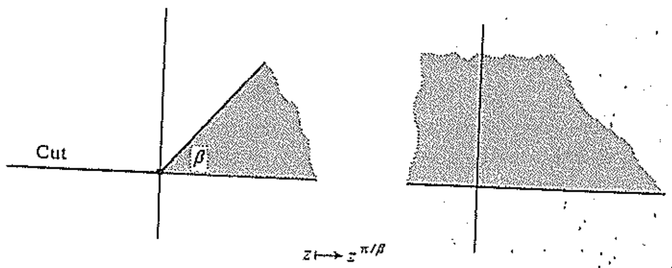


Fig. 10.6

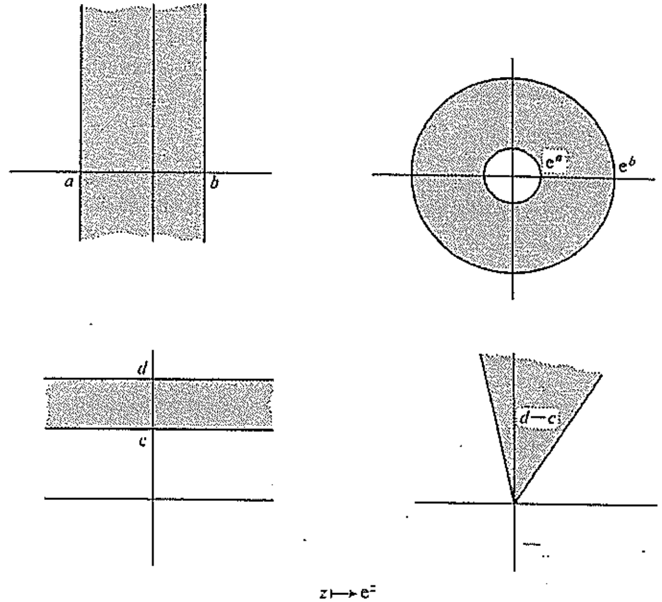


Fig. 10.7

similar families. As an example of a mapping of a different type we consider the simplest form of Joukowski transformation,

$$z\mapsto w = \tfrac{1}{2}(z+z^{-1}).$$

This is given equivalently by

$$\frac{w+1}{w-1} = \left(\frac{z+1}{z-1}\right)^2.$$

It is holomorphic except at 0 and ∞, and conformal except at ±1, where angles are doubled.

Suppose $w = u+iv$ is the image of $z = re^{i\theta}$, so that

$$u = \tfrac{1}{2}(r+r^{-1})\cos\theta,\ v=\tfrac{1}{2}(r-r^{-1})\sin\theta.$$

Then the image of the circle $|z|=\rho$ is the ellipse

$$\frac{u^2}{\frac{1}{4}(\rho+\rho^{-1})^2}+\frac{v^2}{\frac{1}{4}(\rho-\rho^{-1})^2}=1,$$

while the image of the half-line $\arg z = \mu$ is

$$\frac{u^2}{\cos^2\mu} - \frac{v^2}{\sin^2\mu} = 1,$$

which is a hyperbola.

Examples on building conformal mappings

This section gives examples to show how we can combine the mappings introduced above to map an assortment of regions onto simpler ones such as discs and half-planes.

10.18 Example

To find a conformal mapping of the semicircular region $G = \{z : \operatorname{Im} z > 0, |z| < 1\}$ onto $D(0; 1)$.

Solution.

Stage 1 By 10.14, we can express G as

$$\left\{z : \tfrac{1}{2}\pi < \arg\left(\frac{z-1}{z+1}\right) < \pi\right\}$$

(note that G is bounded by arcs through ± 1 subtending angles $\frac{1}{2}\pi$ and 0). Define $g(z) = (z-1)/(z+1) = \zeta$. Under g, G is mapped conformally to the quadrant $Q = \{\zeta : \frac{1}{2}\pi < \arg \zeta < \pi\}$.

Stage 2 Let $\tau = \zeta^2$. Under $\zeta \mapsto \tau$, Q is mapped conformally to the open lower half-plane $\{\tau : \pi < \arg \tau < 2\pi\}$. (Here the non-conformality of $\zeta \mapsto \zeta^2$ at $\zeta = 0$ $(\notin Q)$ works to our advantage.)

Stage 3 The open lower half-plane is $\{\tau : |\tau + i| < |\tau - i|\}$ and so is mapped to $\{w : |w| < 1\}$ under $\tau \mapsto (\tau + i)/(\tau - i) = w$.

A suitable map f can now be seen to be

$$f : z \mapsto w = \frac{(z-1)^2 + i(z+1)^2}{(z-1)^2 - i(z+1)^2} = i\left(\frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}\right);$$

this maps G onto $D(0; 1)$ by construction and, as the composite of conformal maps, is conformal. □

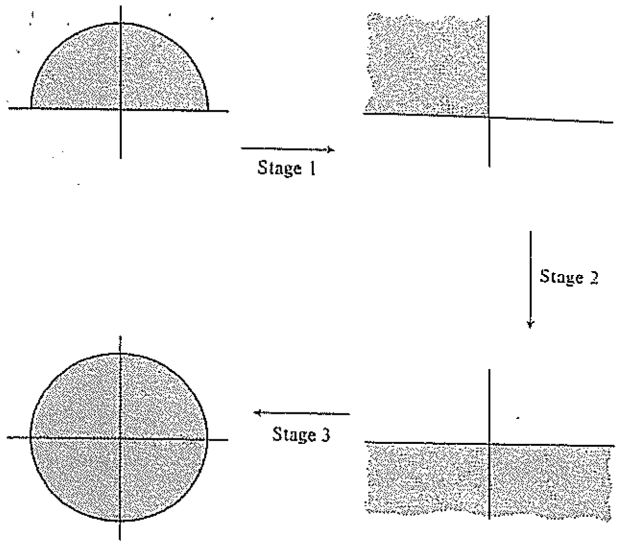


Fig. 10.8

10.19 Example

To find a conformal mapping of the semi-infinite strip $H = \{z : \operatorname{Im} z > 0, 0 < \operatorname{Re} z < \pi\}$ onto a half-plane.

Solution.

Stage 1 Put $\zeta = e^{iz}$. Then $|\zeta| = e^{-\operatorname{Im} z}$ and $\arg \zeta = \operatorname{Re} z \pmod{2\pi}$. So H is mapped by $z \mapsto \zeta$ conformally onto

$$G = \{\zeta : 0 < |\zeta| < 1, 0 < \arg \zeta < \pi\}.$$

Stage 2 We could now proceed as in Stages 1 and 2 of Example 10.18. More directly, we can use the Joukowski transformation $\zeta \mapsto w = \frac{1}{2}(\zeta + \zeta^{-1})$, which is conformal in G. The region G is the union, over $0 < r < 1$, of the semicircular arcs $\zeta = re^{i\theta}$ $(0 < \theta < \pi)$; see Fig. 10.9. Each of these arcs is mapped to an elliptic arc

$$\frac{u^2}{\frac{1}{4}(r + r^{-1})^2} + \frac{v^2}{\frac{1}{4}(r - r^{-1})^2} = 1, \quad v < 0 \quad (\zeta = u + iv).$$

The union of these images covers the open lower half-plane, onto which H is thus mapped by the composite transformation

$$z \mapsto w = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z.$$

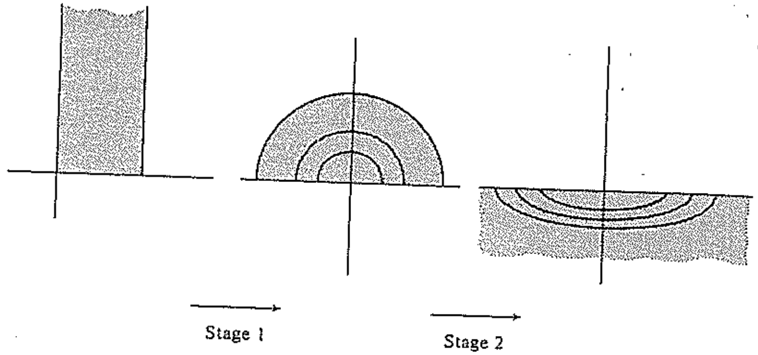


Fig. 10.9

10.20 Example

To find a conformal mapping of the region G exterior to both the circles $|z \pm 1| = \sqrt{2}$ onto the region $\hat{G}$ exterior to the unit circle.

Solution.

Stage 1 The region G is bounded by circular arcs meeting orthogonally at $\pm$i (see Fig. 10.10). Take

$$g : z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}} = \zeta;$$

g is conformal except at $-\mathrm{i} \notin G$. The boundary arcs of G are mapped to half-lines meeting at $g(\mathrm{i}) = 0$, and G is mapped onto a sector S of angle $3\pi/2$ (see 10.7 and 10.14). By conformality $g(0) = -1$ must lie on the bisector of the complementary sector, so S is as shown in Fig. 10.10. Note how we have avoided having to compute the angles subtended by the boundary arcs of G.

Stage 2 Working from the opposite end, we can realize $\hat{G} = \{w : |w| > 1\}$ as the image under the conformal map

$$h : \tau \mapsto \frac{\tau+1}{\tau-1} = w$$

of the right half-plane $\hat{S} = \{\tau : |\tau - 1| < |\tau + 1|\}$.

Stage 3 To transform S onto $\hat{S}$ we seek to multiply angles at 0 by a factor of $\frac{2}{3}$. Formally $\zeta \mapsto \tau = \zeta^{\frac{2}{3}}$ provides the map we want, but

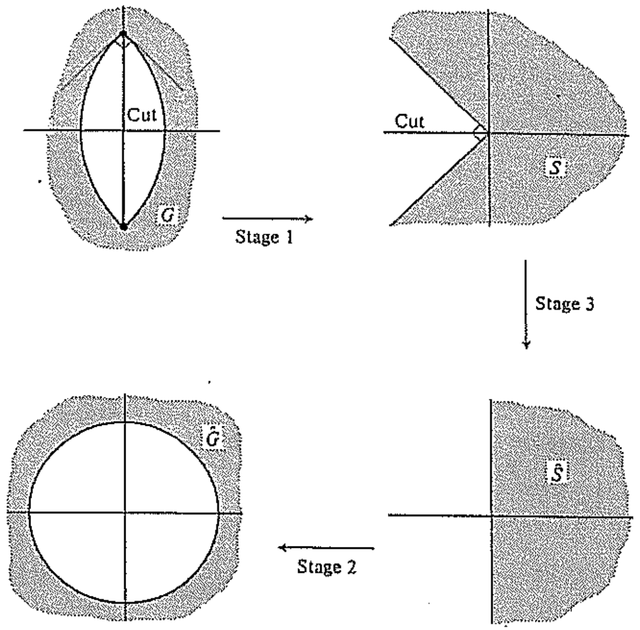


Fig. 10.10

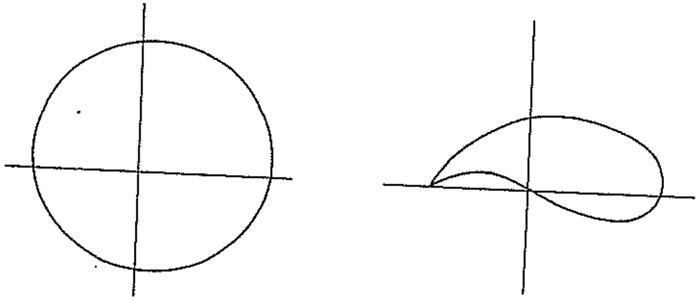
we have to take care because this is a multifunction. We start with the z-plane cut along $[-\mathrm{i}, \mathrm{i}]$. In this cut plane there exists a holomorphic branch k of $[[(z-\mathrm{i})/(z+\mathrm{i})]^{\frac{2}{3}}]$. The map we finally require is $f = h \circ k$. This sends z to w where $[(z-\mathrm{i})/(z+\mathrm{i})]^2 = \tau^3$ and $(\tau+1)/(\tau-1) = w$. Hence $f(z) = w$, where

$$\left(\frac{z-\mathrm{i}}{z+\mathrm{i}}\right)^2 = \left(\frac{w+1}{w-1}\right)^3. \qquad \square$$

10.21 Remarks

Because of the way they act on circlines, Möbius transformations, powers, and exponentials are of most use for mapping regions whose boundaries are made up of circular arcs, lines, and line segments. However their scope is, even so, limited. For example, to handle regions with polygonal boundaries, one must introduce the *Schwarz-Christoffel transformation*, which is awkwardly defined by an integral:

$$z \mapsto \int_0^z (\zeta - z_1)^{-k_1} (\zeta - z_2)^{-k_2} \ldots (\zeta - z_n)^{-k_n}\, \mathrm{d}\zeta.$$



$z \mapsto \frac{1}{2}(z+z^{-1})$

Fig. 10.11

For suitable z_j and k_j and the integrand a suitably specified holomorphic branch, this can be shown to map an open disc onto a region bounded by an n-gon.

Variants on the Joukowski transformation introduced in 10.17 are useful in elementary mapping problems of the type considered already, but also have a particular feature which makes them of interest in fluid dynamics: they map certain circles to (crude models of) aerofoil shapes (see Fig. 10.11). This enables the lift on a model of an aircraft wing to be estimated.

It is by no means obvious that it is possible to construct a conformal mapping from a region with a complicated, spiky boundary onto a civilized region such as D(0; 1), or vice versa. The definitive theorem on conformal mapping, which we state without proof, is thus very striking.

The Riemann mapping theorem Let G be a simply connected region with $G \neq \mathbb{C}$. Then there exists a one-to-one conformal mapping f from G onto D(0; 1) with $f^{-1}: \mathrm{D}(0;1) \to G$ also conformal.

It is worth noting that in each of our worked examples, the function we defined not only took one prescribed region, G, onto another, $\hat{G}$, but also mapped the boundary of G onto the boundary of $\hat{G}$. This extension of a conformal mapping to the boundary of a region is needed in many applications (see 10.39). In general, whether it is possible depends on the topological nature of the boundary.

Holomorphic mappings: some theory

It is often necessary to construct a conformal mapping $f: G \to \hat{G}$ such that the inverse mapping $f^{-1}: \hat{G} \to G$ exists and is also conformal; see 10.38. We present a group of theorems which have a bearing on this problem and are of independent interest. Since there are common themes in the proofs we begin with some general remarks.

10.22 Observations

Suppose G is open, $f \in \mathrm{H}(G)$, and $a \in G$.

(1) Assume that G is a region and f non-constant. Then $f - f(a)$ is never zero in some $\mathrm{D}'(a; r)$ (by the Identity theorem, 5.14).

(2) Let f be one-to-one. Then f' has isolated zeros (by Stage 1 of the Identity theorem proof, applied to f'; see 5.14).

(3) Choose r such that $\bar{\mathrm{D}}(a; r) \subseteq G$ and suppose $f - f(a)$ is non-zero on γ^*, where $\gamma = \gamma(a; r)$. Let $m := \inf\{|f(z) - f(a)| : z \in \gamma^*\}$. Then

(i) $m > 0$ (by 1.18);

(ii) for each $w \in \mathrm{D}(f(a); m)$, $f - f(a)$ and $f - w$ have the same number of zeros in $\mathrm{D}(a; r)$ (counted according to multiplicity) (by Rouché's theorem, 7.7: for $z \in \gamma^*$, $|f(z) - f(a)| \geqslant m > |f(a) - w| = |f(a) - f(z) + f(z) - w|$).

10.23 Theorem

Suppose f is holomorphic and one-to-one in an open set G. Then f is conformal in G.

Proof. Assume for a contradiction that there exists $a \in G$ such that $f'(a) = 0$. Choose r such that $\bar{\mathrm{D}}(a; r) \subseteq G$ and f' is never zero in $\mathrm{D}'(a; r)$. This is possible by 10.22(2). Let $w \in \mathrm{D}'(f(a); m)$, where m is as in 10.22(3). By 10.22(3)(ii), $f - f(a)$ and $f - w$ have the same number of zeros in $\mathrm{D}(a; r)$. The function $f - f(a)$ has a zero of order at least two at a (by 6.8). On the other hand, $f - w$ cannot have two distinct zeros, since f is one-to-one, and cannot have a zero of order greater than one, since $f - w$ and $(f - w)'$ cannot both be zero at any point in $\mathrm{D}(a; r)$. □

10.24 The Open mapping theorem

Suppose f is holomorphic and non-constant in an open set G. Then $f(G)$ is open.