6. Let $z=re^{i\theta}$ and $w=Re^{i\phi}$, where $0\leqslant r<R$. Prove that

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right)=\frac{|w|^2-|z|^2}{|w-z|^2}=\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}.$$

(These formulae for the *Poisson kernel* are needed in Chapter 10.)

7. The usual order relation $>$ on $\mathbb{R}$ satisfies
(i) $x\neq 0 \Rightarrow x>0$ or $-x>0$, but not both, and
(ii) $x, y>0 \Rightarrow x+y>0$ and $xy>0$.
Show that there does not exist a relation $>$ on $\mathbb{C}$ satisfying (i) and (ii). (Hint: consider i.)

8. Describe the following sets geometrically.
(i) $\{z: 1<\operatorname{Im}(z+i)<2\}$,
(ii) $\{z: |z+2i|\geqslant 2\}$,
(iii) $\{z: |z-i|<|z-1|\}$,
(iv) $\{z=|z|e^{i\theta}: -\pi<\theta<\pi/2\}$.

9. Prove that any punctured disc $D'(a;r)$ is an open set. What are its limit points? What is its closure?

10. Let G be an open set in $\mathbb{C}$. Which of the following sets are open:
(i) $\{z: \bar{z}\in G\}$, (ii) $\{\operatorname{Re} z: z\in G\}$, (iii) $\{z: z\in G \text{ or } \bar{z}\in G\}$,
(iv) $\{z\in G: \operatorname{Im} z>0\}$?

11. Describe geometrically each of the following sets. Which are open, which are closed, and which are compact? Find the closures of the non-closed sets.
(i) $\{z: |z-1-i|=1\}$,
(ii) $\{z: |z-1+i|\geqslant |z-1-i|\}$,
(iii) $\{z: |z+i|\neq |z-i|\}$,
(iv) $\{z=|z|e^{i\theta}: \frac{1}{4}\pi<\theta\leqslant\frac{3}{4}\pi\}$,
(v) $\{z: \operatorname{Re} z<1 \text{ or } \operatorname{Im}(z-1)\neq 0\}$,
(vi) $\{z: |z-1|<1, |z|=|z-2|\}$,
(vii) $\{z: |z-2|>3, |z|<2\}$,
(viii) $\{z: |z^2-1|<1\}$,
(ix) $\{z: |z|^2>z+\bar{z}\}$,
(x) $\{z: \operatorname{Im}[(z+i)/2i]<0\}$.

12. Which of the following complex sequences converge:

(i) $\left\langle \frac{1}{n} i^n \right\rangle$, (ii) $\left\langle \frac{(-1)^n n}{n+i} \right\rangle$, (iii) $\left\langle \frac{n^2+in}{n^2+i} \right\rangle$?

13. Suppose $\langle z_n\rangle$ is a complex sequence of distinct points converging to z. Prove that z is the unique limit point of the set $\{z_n: n=1, 2, \ldots\}$.

14. For each of the following choices of $f(z)$, either obtain $\lim_{z\to 0} f(z)$ or prove that the limit fails to exist:

(i) $\dfrac{|z|^2}{z}$, (ii) $\dfrac{\bar{z}}{z}$, (iii) $\dfrac{z+1}{|z|-1}$, (iv) $\dfrac{(\operatorname{Re} z)(\operatorname{Im} z)}{|z|}$.

15. Prove that f is continuous on $\mathbb{C}$ when (i) $f(z)=\bar{z}$, (ii) $f(z)=\operatorname{Im} z$, (iii) $f(z)=\operatorname{Re} z^2$.

16. Define a function f by $f(z)=z/(1+|z|)$. Show that f is continuous and that it maps $\mathbb{C}$ one-to-one onto $D(0;1)$.

2 Holomorphic functions and power series

Chapter 1 did not go far enough to reveal the true flavour of complex analysis. Continuous functions play only an ancillary and technical role in the subject. Much more important are the holomorphic functions which this chapter introduces. Loosely, holomorphic means differentiable. The formal definition of holomorphy, given in 2.2, is restrictive enough to lead to powerful and elegant theorems, yet wide enough to allow a wealth of practical applications. Power series are central to the development of the theory. As Theorems 2.12 and 5.9 will show, they define, and can be used to represent, holomorphic functions.

That part of Chapter 10 dealing with examples of holomorphic mappings can profitably be studied immediately after this chapter.

Holomorphic functions

2.1 Functions

In Chapter 1 we took for granted the concept of a function. Formally, given $S\subseteq\mathbb{C}$, a mapping $f: S\to\mathbb{C}$ which assigns to each $z\in S$ a unique complex number $f(z)$ is called a *complex-valued function*, or simply a *function*, on S. Reference back to this fundamental definition is not merely belated pedantry. This is simply an opportune point at which to emphasize the inherent 'one-valuedness' of a function. (Later we deal also with what we call multifunctions: a *multifunction* is a rule assigning a subset of $\mathbb{C}$ (finite or infinite) to each element of its domain set S.)

Strictly, given a function f, we should distinguish between f (the mapping) and $f(z)$ (the image of the point z under f). However, where it would be cumbersome to do otherwise, we allow $f(z)$ to denote the function and write, for example 'z^2' in place of 'the function f defined by $f(z)=z^2$'. We also adopt the notation $z\mapsto w=f(z)$ to indicate that z is mapped by f to w.

2.2 Definitions

(1) A function f defined on an open subset G of $\mathbb{C}$ is *differentiable* at $z \in G$ if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists. When the limit exists it is denoted by $f'(z)$ or df/dz.
(2) A function which is differentiable at every point of an open set G is *holomorphic* in G. The set of functions holomorphic in G is denoted by $H(G)$.
(3) If S is any subset of $\mathbb{C}$, f is holomorphic in S if $f \in H(G)$ for some open set G containing S.

2.3 Remarks

(1) Suppose f is differentiable at a point z of an open set G in which f is defined. Then, for $z+h \in G$,

$$f(z+h)=f(z)+hf'(z)+h\varepsilon(h) \quad \text{where} \quad \varepsilon(h)\to 0 \ \text{as} \ h\to 0.$$

This is immediate on writing, for $h \neq 0$, $\varepsilon(h)$ for

$$\frac{f(z+h)-f(z)}{h}-f'(z),$$

and shows that $f(z+h)\to f(z)$ as $h\to 0$. Hence differentiability of f at z implies continuity of f at z. We have established the technically useful fact that, if f is holomorphic in a set S, then f is continuous on S.

(2) Note the role played by open sets in 2.2. Provided G is open, whenever $z \in G$ there exists $r>0$ such that $z+h \in G$ for all h with $|h|<r$. This has the effect that in the computation of the limit defining $f'(z)$, $z+h$ is free to approach z from any direction as $h\to 0$. For f to be differentiable at z, the value of the limit must be independent of the manner in which $h\to 0$. We exploit this in the derivation of the following result by equating the expressions obtained for $f'(z)$ on taking (i) h real and (ii) h purely imaginary.

2.4 Theorem (the Cauchy–Riemann equations)

Let f be defined on an open set G and be differentiable at $z=x+iy \in G$. Let

$$f(z)=u(x,y)+iv(x,y),$$

where u and v are real-valued functions on G (regarded as a subset of $\mathbb{R}^2$). Then u and v have first order partial derivatives at (x,y) (denoted u_x, u_y, v_x, v_y) and these satisfy the *Cauchy–Riemann equations*

$$u_x=v_y, \qquad u_y=-v_x.$$

Proof. From Definition 2.2(1),

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}.$$

Hence

$$f'(z)=\lim_{\substack{h\to 0\\ h\in\mathbb{R}}}\left(\frac{u(x+h,y)-u(x,y)}{h}+i\frac{v(x+h,y)-v(x,y)}{h}\right)=u_x+iv_x$$

and

$$f'(z)=\lim_{\substack{h\to 0\\ h=ik\\ k\in\mathbb{R}}}\left(\frac{u(x,y+k)-u(x,y)}{ik}+\frac{v(x,y+k)-v(x,y)}{k}\right)=\frac{1}{i}u_y+v_y.$$

Equating the two expressions for $f'(z)$ gives

$$u_x+iv_x=-iu_y+v_y.$$

Equating real and imaginary parts we obtain

$$u_x=v_y, \qquad u_y=-v_x. \qquad \square$$

2.5 Example

Let $f(z)=|z|$ on $G=\mathbb{C}$. Prove f is not differentiable at any point.

Solution. In the notation of Theorem 2.4,

$$u(x,y)=(x^2+y^2)^{\frac{1}{2}} \quad \text{and} \quad v(x,y)=0.$$

Then $v_x=v_y=0$ and for $(x,y)\neq(0,0)$,

$$u_x=x(x^2+y^2)^{-\frac{1}{2}}, \qquad u_y=y(x^2+y^2)^{-\frac{1}{2}}.$$

The Cauchy–Riemann equations fail to hold, and f to be differentiable, at any point $z\neq 0$. The point 0 requires separate attention. From first principles,

$$\frac{f(h)-f(0)}{h}=\frac{|h|}{h}\to\begin{cases}1 & \text{as } h\to 0 \quad (h \text{ real and positive}),\\ -1 & \text{as } h\to 0 \quad (h \text{ real and negative}).\end{cases}$$

Hence $f'(0)$ does not exist. $\square$

The Cauchy–Riemann equations are useful for proving non-differentiability. They are *not*, on their own, a sufficient condition for differentiability (see Exercise 2.3). we return to this point in Chapter 10. There we also use the Cauchy–Riemann equations to establish that the real and imaginary parts of a holomorphic function satisfy Laplace's equation in two dimensions. This fact forms the basis of many of the applications of complex analysis to physical problems.

The Cauchy–Riemann equations also have theoretical applications, as the next result shows.

2.6 Proposition

Suppose $f \in H(D(0; R))$. Then
(1) if $f' = 0$ in $D(0; R)$, f is constant;
(2) if $|f|$ is a constant, c, in $D(0; R)$, f is constant.

Proof. We adopt the notation of Theorem 2.4. The proof of this theorem shows that for $z = x + iy \in D(0; R)$,

$$f'(z) = u_x + iv_x = -iu_y + v_y.$$

Suppose $f' = 0$. Then $u_x = v_x = u_y = v_y = 0$. Fix arbitrary points $p = a + ib$ and $q = c + id \in D(0; R)$. We shall show that $f(p) = f(q)$. At least one of $s = c + ib$ and $t = a + id$ lies in $D(0; R)$; without loss of generality suppose s does. Each of $x \mapsto u(x, b)$ and $y \mapsto u(c, y)$ is a real-valued function of a real variable with zero derivative and so is constant, by the Mean-value theorem. Hence

$$u(a, b) = u(c, b), \qquad u(c, b) = u(c, d),$$

and similarly

$$v(a, b) = v(c, b), \qquad v(c, b) = v(c, d).$$

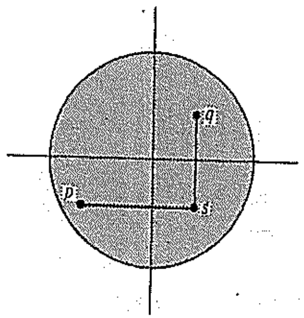

Fig. 2.1

We conclude that $f(p) = f(s) = f(q)$.

Now suppose that $|f| = c$, so that $u^2 + v^2 = c^2$. Then

$$uu_x + vv_x = 0, \qquad uu_y + vv_y = 0,$$

whence, by the Cauchy–Riemann equations,

$$uu_x - vu_y = 0, \qquad uu_y + vu_x = 0.$$

Elimination of u_y gives $0 = (u^2 + v^2)u_x = c^2 u_x$. If $c = 0$, f is trivially constant. Otherwise, $u_x = 0$, and similarly u_y, v_x, and v_y are zero. We deduce, as above, that f is constant. □

This proof is unaesthetic, but instructive. It provides a stepping stone between complexified real analysis and complex analysis proper. It motivates the introduction, in Chapter 3, of connected sets; connectedness is the characteristic of $D(0; R)$ germane to the proof of Proposition 2.6; see Proposition 3.18.

2.7 Holomorphic functions: elementary properties and examples

We have so far refrained from giving examples of holomorphic functions because except in the simplest cases it is very laborious to check holomorphy from the definition. (As in real analysis, proving differentiability from first principles is useful mainly as an exercise in evaluating limits.) Instead, we build up a catalogue of holomorphic functions by forming products, composites, etc.

The following properties are stated for functions holomorphic in an arbitrary set S. They are proved by checking the appropriate differentiability conditions, pointwise on a suitable open set $G \supseteq S$ (see Definition 2.2(3)). We omit the details as the proofs are formally identical to their real counterparts.

(1) Let f and g be holomorphic in S and $\lambda \in \mathbb{C}$. Then λf, $f + g$, and fg (defined pointwise in the usual way) are holomorphic in S and the usual differentiation rules apply: for all $z \in S$,

$$(\lambda f)'(z) = \lambda f'(z), \qquad (f+g)'(z) = f'(z) + g'(z),$$
$$(fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

(2) **The chain rule** Let f be holomorphic in S and let g be holomorphic in $f(S)$. Then the composite function $g \circ f$, given by $(g \circ f)(z) = g[f(z)]$, is holomorphic in S and, for all $z \in S$,

$$(g \circ f)'(z) = g'[f(z)]f'(z).$$