10 Conformal mapping and harmonic functions

The first part of the chapter concerns angle-preserving mappings between regions in the complex plane. Such mappings are of intrinsic geometric interest and are important in advanced complex analysis. They are also worth studying because of their usefulness in solving problems concerning harmonic functions in $\mathbb{R}^2$ (in particular, certain two-dimensional problems in fluid dynamics). The final section reveals the striking parallels between the theory of holomorphic functions and that of harmonic functions.

We shall adopt a more cavalier attitude to the argument of a complex number than previously. This is legitimate since we are concerned with the determination of argument 'a point at a time' and not with the variation of the argument of a moving point. We accordingly write $\arg z$ to denote any choice from $[\arg z] = \{\theta : z = |z| e^{i\theta}\}$. The price to pay is that some equations only hold modulo an integer multiple of 2π.

It will sometimes be convenient to work in the extended plane $\tilde{\mathbb{C}}$ introduced in 6.13. We adopt the following conventions:

$$a \pm \infty = \pm\infty + a = \infty, \quad a/\infty = 0 \quad \text{for all } a \in \mathbb{C},$$
$$a \times \infty = \infty \times a = \infty, \quad a/0 = \infty \quad \text{for all } a \in \mathbb{C}\setminus\{0\},$$
$$\infty + \infty = \infty \times \infty = \bar{\infty} = \infty.$$

Circles and lines revisited

Any circle is given by an equation $|z-a| = r$. However our repertoire of techniques for solving mapping problems involving circles will be greatly enlarged by having another form of equation available.

We work in $\tilde{\mathbb{C}}$, and adjoin ∞ to any line in $\mathbb{C}$. Then circles and straight lines in $\tilde{\mathbb{C}}$ correspond to circles on the Riemann sphere Σ, with straight lines associated with circles passing through the north pole. The distinction between circles and lines is thus rather an artificial one, and we introduce the generic name *circline* to cover both.

10.1 Definition

Points α and β are *inverse with respect to the circle* $|z-a| = r$ if $(\alpha - a)\overline{(\beta - a)} = r^2$ (we include the pair $\alpha = a$, $\beta = \infty$); see Fig. 10.1 below. Note that α, β, and a are collinear. Points α and β are *inverse with respect to a straight line* ℓ if β is the reflection of α in ℓ.

10.2 Theorem (Inverse point representation of circlines)

Let α and β belong to $\mathbb{C}$, $\alpha \neq \beta$. Then for any $\lambda > 0$, the equation $\left|\dfrac{z-\alpha}{z-\beta}\right| = \lambda$ represents a circline with inverse points α and β, and every such circline can be so represented.

Proof. Consider the equation $\left|\dfrac{z-\alpha}{z-\beta}\right| = \lambda$. Denote α by A, β by B, and the variable point z by P. If $\lambda = 1$, the locus of P is the perpendicular bisector of AB; it has α and β as inverse points.

Now assume $\lambda \neq 1$. The equation gives the locus of points P for which the ratio $AP:PB$ has the constant value λ. The locus is a circle (known as the circle of Apollonius). This can be proved geometrically, but it is simpler to switch to cartesian coordinates (a strategy usually to be avoided in complex analysis): Put $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, and $z = x + iy$. The equation becomes

$$\left(x - \frac{\alpha_1 - \lambda^2\beta_1}{1-\lambda^2}\right)^2 + \left(y - \frac{\alpha_2 - \lambda^2\beta_2}{1-\lambda^2}\right)^2 = K,$$

where K is a constant, and this certainly represents a circle. Define z_1 and z_2 by

$$\alpha - z_1 = \lambda(z_1 - \beta) \quad \text{and} \quad \alpha - z_2 = \lambda(\beta - z_2).$$

These points lie on the circle and are collinear with α and β, so they are endpoints of a diameter (see Fig. 10.1). The circle has centre $a = \frac{1}{2}(z_1 + z_2)$ and radius $r = \frac{1}{2}|z_1 - z_2|$. Adding and subtracting the equations above gives

$$\alpha - a = \tfrac{1}{2}\lambda(z_1 - z_2) \quad \text{and} \quad 2\lambda(\beta - a) = (z_1 - z_2).$$

Hence $(\alpha - a)\overline{(\beta - a)} = \frac{1}{4}(z_1 - z_2)\overline{(z_1 - z_2)} = r^2$, which proves that α and β are inverse points with respect to the circle.

Conversely, it is clear that the equation of any line can be written as $|z-\alpha| = |z-\beta|$ for some α and β. Also, given any circle $|z-a| = r$, we can choose $\alpha \neq a$ and then $\beta = a + r^2\overline{(\alpha - a)}^{-1}$ to make α and β inverse points. Take any point z_0 on the circle and

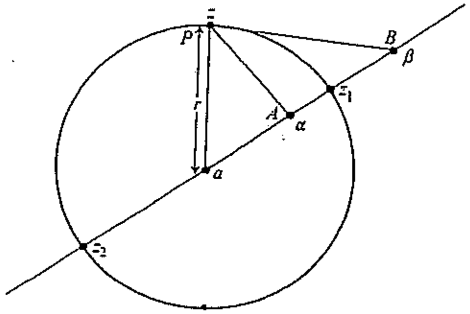

Fig. 10.1

let $\lambda = |z_0 - \alpha|/|z_0 - \beta|$. Then the circle $|(z-\alpha)/(z-\beta)| = \lambda$ coincides with the given one. □

Note A circline is uniquely determined by a pair of inverse points and a point on the circline.

10.3 The unit circle ($|z|=1$)

Any points α and $1/\bar{\alpha}$ in $\mathbb{C}$ are inverse with respect to the unit circle, and the point 1 lies on the circle. Hence, by Theorem 10.2, the equation can be written

$$\left|\frac{z-\alpha}{z-1/\bar{\alpha}}\right| = \left|\frac{1-\alpha}{1-1/\bar{\alpha}}\right|, \quad \text{that is,}$$

$$\left|\frac{z-\alpha}{\bar{\alpha}z-1}\right| = 1 \quad (\text{since } |\alpha-1| = |\bar{\alpha}-1|).$$

10.4 Representation of circular arcs

If P is a variable point on a circular arc with endpoints A and B, then $\mu = \angle APB$ is constant. From Fig. 10.2, $\mu = \theta - \phi$, where $\arg(z-\alpha) = \theta$ and $\arg(z-\beta) = \phi$. Hence the arc APB has equation

$$\arg(z-\alpha) - \arg(z-\beta) = \mu \pmod{2\pi},$$

that is,

$$\arg\left(\frac{z-\alpha}{z-\beta}\right) = \mu \pmod{2\pi} \qquad (z \neq \alpha, \beta).$$

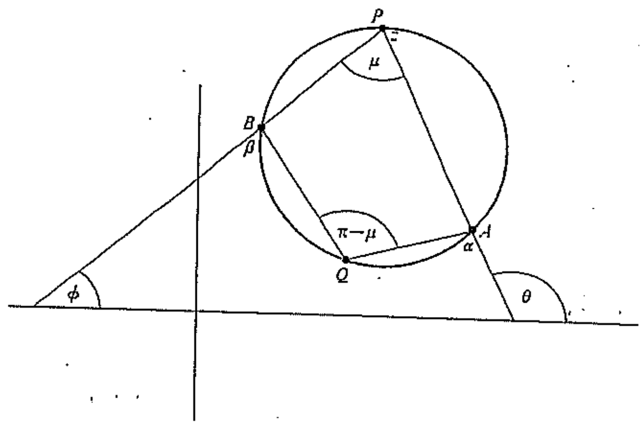

Fig. 10.2

Similarly the equation of the arc AQB is (note the signs!)

$$\arg\left(\frac{z-\alpha}{z-\beta}\right) = -(\pi-\mu) \pmod{2\pi} \qquad (z \neq \alpha, \beta).$$

10.5 Coaxal circles

For fixed α and β we have, as λ and μ vary, two families of circles:
(1) $C_1(\alpha, \beta)$: circles

$$\left|\frac{z-\alpha}{z-\beta}\right| = \lambda,$$

having α and β as inverse points, and
(2) $C_2(\alpha, \beta)$: circles

$$\arg\left(\frac{z-\alpha}{z-\beta}\right) = \begin{cases}\mu \\ -(\pi-\mu)\end{cases} \pmod{2\pi},$$

through α and β.

Each of the families is said to form a *coaxal system*. These systems of circles have interesting geometric properties. It can be shown, for example, that any member of $C_1(\alpha, \beta)$ cuts any member of $C_2(\alpha, \beta)$ orthogonally.

Möbius transformations, exponentials, and powers discussed below.

Möbius transformations

10.9 Definition

A Möbius transformation is a mapping of the form

$$f: z \mapsto \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{C},\ ad-bc \neq 0).$$

(The excluded case $ad-bc=0$ produces a constant mapping.)

The Möbius transformation f is best viewed as a mapping of $\tilde{\mathbb{C}}$ to itself, with (by definition) $f(-d/c)=\infty$, $f(\infty)=a/c$. The map $f:\tilde{\mathbb{C}}\to\tilde{\mathbb{C}}$ is one-to-one and onto, with inverse

$$f^{-1}: w \mapsto \frac{dw-b}{a-cw}$$

also a Möbius transformation. It is easily checked that the Möbius transformations form a group under the operation of composition of maps. A general Möbius transformation can be built up from mappings of the following types:

$z \mapsto ze^{i\phi}$ (ϕ real)	(anticlockwise rotation through ϕ),
$z \mapsto Rz$ ($R>0$)	(stretching by a factor of R),
$z \mapsto z+a$ ($a\in\mathbb{C}$)	(translation by a),
$z \mapsto 1/z$	(inversion).

Suppose $f(z)=(az+b)/(cz+d)$. Then $f'(z)=(ad-bc)/(cz+d)^2$, which shows that f is conformal on $\mathbb{C}\setminus\{-d/c\}$. Theorem 10.10 shows just how much room for manoeuvre we have in constructing Möbius transformations.

10.10 Theorem

Suppose each of $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ is a triple of distinct points in $\tilde{\mathbb{C}}$. Then there exists a unique Möbius transformation f such that $f(z_k)=w_k$ $(k=1,2,3)$, given by $f: z\mapsto w=f(z)$, where

$$\left(\frac{w-w_1}{w-w_3}\right)\left(\frac{w_2-w_3}{w_2-w_1}\right)=\left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right).$$

Proof. The map

$$g: z\mapsto \left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right)$$

takes z_1, z_2, and z_3 to $0, 1$, and ∞, respectively. Construct h in the same way as g, to map w_1, w_2, and w_3 to $0, 1$, and ∞. Then $f=h^{-1}\circ g$ is the map in the statement of the theorem and $f(z_k)=w_k (k=1,2,3)$.

For uniqueness it is enough to show that the only Möbius transformation $f: z\mapsto (az+b)/(cz+d)$ fixing 0, 1, and ∞ is the identity map. The conditions $f(0)=0$, $f(\infty)=\infty$, and $f(1)=1$ force in turn $b=0$, $c=0$, and $a=d$, so that $f(z)=z$ for all z. □

One can show that Möbius transformations map circlines to circlines by using any of the standard representations of circles and lines. The proof below shows more: the preservation of inverse points.

10.11 Theorem

Let S be a circline with inverse points α and β $(\alpha,\beta\in\mathbb{C},\ \alpha\neq\beta)$ and let f be a Möbius transformation. Then f maps S to a circline with inverse points $f(\alpha)$ and $f(\beta)$.

Proof. We use Theorem 10.2 to write the equation of S in the form $|(z-\alpha)/(z-\beta)|=\lambda$. Suppose $w=f(z)=(az+b)/(cz+d)$, so $z=(dw-b)/(a-cw)$. Then

$$\left|\frac{(\alpha c+d)w-(\alpha a+b)}{(\beta c+d)w-(\beta a+b)}\right|=\lambda.$$

We may rewrite this as

(i) $\left|\dfrac{w-f(\alpha)}{w-f(\beta)}\right|=\lambda\left|\dfrac{\beta c+d}{\alpha c+d}\right|$, if $\alpha c+d\neq 0$ and $\beta c+d\neq 0$, or

(ii) $|w-f(\alpha)|=\lambda\left|\dfrac{\beta a+b}{\alpha c+d}\right|$, if $\alpha c+d\neq 0$ and $\beta c+d=0$, or

(iii) $|w-f(\beta)|=\dfrac{1}{\lambda}\left|\dfrac{\alpha a+b}{\beta c+d}\right|$, if $\alpha c+d=0$ and $\beta c+d\neq 0$.

(Note that $\alpha c+d$ and $\beta c+d$ cannot both be zero.) In each case the image of S is a circline with $f(\alpha)$ and $f(\beta)$ as inverse points; in cases (ii) and (iii) the images of the original inverse points are the centre of the circle and the point ∞. □

10.12 Theorem

There exists a Möbius transformation mapping any given circline to any other given circline.

Proof. The result follows immediately from Theorems 10.10 and 10.11, since a circline is uniquely determined by three points (these may be taken to be a pair of inverse points and one point on the circline, or three points on it). □

10.13 Examples

(1) To find the images under $f: z \mapsto w = (z-1)^{-1}$ of (a) the real axis, (b) the imaginary axis, (c) the circle centre 0, radius r.

Solution (a) The real axis is mapped to the circline through $f(0) = -1$, $f(1) = \infty$, $f(\infty) = 0$, viz. the real axis. (Alternatively one may use the fact that the real axis has equation $z = \bar{z}$.)
(b) The imaginary axis has equation $|z-1| = |z+1|$ and is mapped to the circline with equation $|2w+1| = 1$, which is the circle with centre $-\frac{1}{2}$ and radius $\frac{1}{2}$.
(c) The required image has equation $|w+1| = r|w|$. If $r = 1$, it is the line $\operatorname{Re} w = -\frac{1}{2}$. If $r \neq 1$, it is a circle with -1 and 0 as inverse points. It has the points $(\pm r - 1)^{-1}$ as the ends of a diameter (see Fig. 10.1), and hence has centre $(r^2-1)^{-1}$ and radius $r|1-r^2|^{-1}$. □
(2) To find all Möbius transformations mapping the unit circle T to itself and mapping α to 0.

Solution. Let f satisfy the required conditions and map 1 to $e^{i\phi} \in \mathrm{T}$. The points α and $1/\bar{\alpha}$ are inverse points with respect to T and so are mapped by f to 0 and ∞ (by 10.11). The unique Möbius transformation taking α, 1, and $0/\bar{\alpha}$ to 1, $e^{i\phi}$, and ∞, respectively, is given by Theorem 10.10 to be $z \mapsto w$ where

$$\frac{w}{e^{i\phi}} = \left(\frac{z-\alpha}{z-1/\bar{\alpha}}\right)\left(\frac{1-1/\bar{\alpha}}{1-\alpha}\right) = \left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)\left(\frac{\bar{\alpha}-1}{1-\alpha}\right).$$

But $|1-\alpha| = |1-\bar{\alpha}|$, so, for some real number ψ,

$$w = e^{i\psi}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right).$$

□

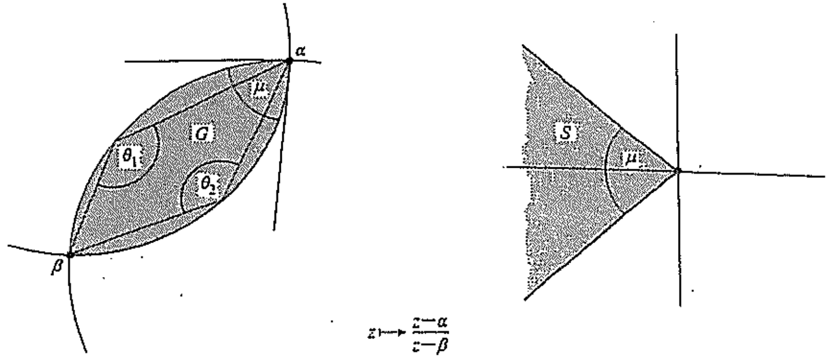

Fig. 10.5

10.14 Mappings of regions bounded by circular arcs

A Möbius transformation can be used, *inter alia*, to map a lozenge-shaped region bounded by circular arcs onto a sector. Let G be as shown in Fig. 10.5. By 10.4,

$$G = \left\{z : \theta_1 < \arg\left(\frac{z-\alpha}{z-\beta}\right) < 2\pi - \theta_2\right\}.$$

The Möbius transformation $f: z \mapsto (z-\alpha)/(z-\beta)$ maps G conformally onto the sector

$$S = \{w : \theta_1 < \arg w < 2\pi - \theta_2\}.$$

Note the effect of mapping β to ∞; circular arcs ending at β transform to half-lines. Observe also that, because f is conformal at α, the angle subtended by S at 0 equals the angle μ between the bounding arcs of G. Note that the image sector is determined completely by the image of a single point other than α or β. By taking the map $z \mapsto k(z-\alpha)/(z-\beta)$, with $k = e^{i\varphi}$ ($\varphi \in \mathbb{R}$) chosen suitably, the image sector can be swung round into any desired position.

This argument deals with mappings of regions bounded by two members of a coaxal system $C_2(\alpha, \beta)$ as defined in 10.5. Exercise 10.8 concerns the mapping of a region bounded by two non-concentric circles, which form members of some coaxal system $C_1(\alpha, \beta)$.

while the image of the half-line $\arg z = \mu$ is

$$\frac{u^2}{\cos^2\mu} - \frac{v^2}{\sin^2\mu} = 1,$$

which is a hyperbola.

Examples on building conformal mappings

This section gives examples to show how we can combine the mappings introduced above to map an assortment of regions onto simpler ones such as discs and half-planes.

10.18 Example

To find a conformal mapping of the semicircular region $G = \{z : \operatorname{Im} z > 0, |z| < 1\}$ onto $D(0; 1)$.

Solution.

Stage 1 By 10.14, we can express G as

$$\left\{z : \tfrac{1}{2}\pi < \arg\left(\frac{z-1}{z+1}\right) < \pi\right\}$$

(note that G is bounded by arcs through ± 1 subtending angles $\frac{1}{2}\pi$ and 0). Define $g(z) = (z-1)/(z+1) = \zeta$. Under g, G is mapped conformally to the quadrant $Q = \{\zeta : \frac{1}{2}\pi < \arg \zeta < \pi\}$.

Stage 2 Let $\tau = \zeta^2$. Under $\zeta \mapsto \tau$, Q is mapped conformally to the open lower half-plane $\{\tau : \pi < \arg \tau < 2\pi\}$. (Here the non-conformality of $\zeta \mapsto \zeta^2$ at $\zeta = 0$ ($\notin Q$) works to our advantage.)

Stage 3 The open lower half-plane is $\{\tau : |\tau + i| < |\tau - i|\}$ and so is mapped to $\{w : |w| < 1\}$ under $\tau \mapsto (\tau + i)/(\tau - i) = w$.

A suitable map f can now be seen to be

$$f : z \mapsto w = \frac{(z-1)^2 + i(z+1)^2}{(z-1)^2 - i(z+1)^2} = i\left(\frac{z^2 + 2iz + 1}{z^2 - 2iz + 1}\right);$$

this maps G onto $D(0; 1)$ by construction and, as the composite of conformal maps, is conformal. □

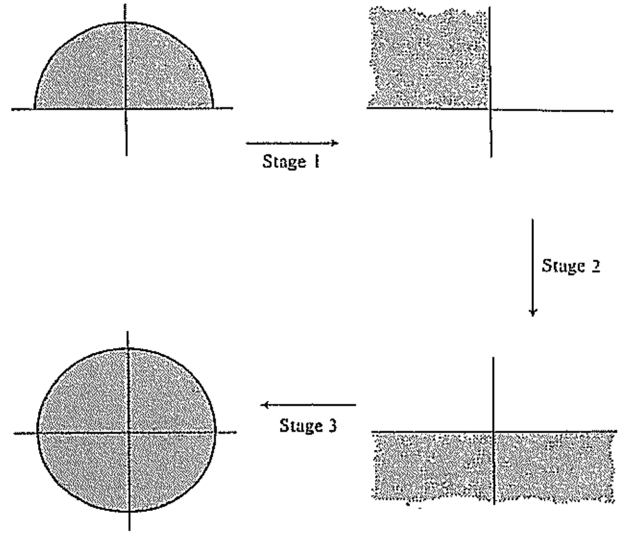

Fig. 10.8

10.19 Example

To find a conformal mapping of the semi-infinite strip $H = \{z : \operatorname{Im} z > 0, 0 < \operatorname{Re} z < \pi\}$ onto a half-plane.

Solution.

Stage 1 Put $\zeta = e^{iz}$. Then $|\zeta| = e^{-\operatorname{Im} z}$ and $\arg \zeta = \operatorname{Re} z \pmod{2\pi}$. So H is mapped by $z \mapsto \zeta$ conformally onto

$$G = \{\zeta : 0 < |\zeta| < 1, 0 < \arg \zeta < \pi\}.$$

Stage 2 We could now proceed as in Stages 1 and 2 of Example 10.18. More directly, we can use the Joukowski transformation $\zeta \mapsto w = \frac{1}{2}(\zeta + \zeta^{-1})$, which is conformal in G. The region G is the union, over $0 < r < 1$, of the semicircular arcs $\zeta = re^{i\theta}$ $(0 < \theta < \pi)$; see Fig. 10.9. Each of these arcs is mapped to an elliptic arc

$$\frac{u^2}{\frac{1}{4}(r + r^{-1})^2} + \frac{v^2}{\frac{1}{4}(r - r^{-1})^2} = 1, \quad v < 0 \quad (\zeta = u + iv).$$

The union of these images covers the open lower half-plane, onto which H is thus mapped by the composite transformation

$$z \mapsto w = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos z.$$

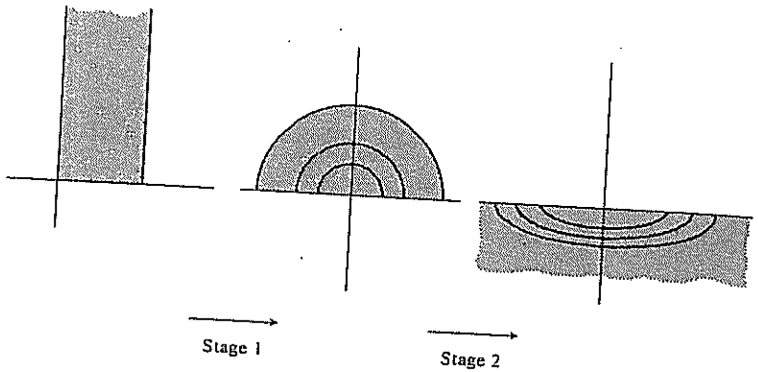

Fig. 10.9

10.20 Example

To find a conformal mapping of the region G exterior to both the circles $|z \pm 1| = \sqrt{2}$ onto the region $\hat{G}$ exterior to the unit circle.

Solution.

Stage 1 The region G is bounded by circular arcs meeting orthogonally at $\pm i$ (see Fig. 10.10). Take

$$g : z \mapsto \frac{z-i}{z+i} = \zeta;$$

g is conformal except at $-i \notin G$. The boundary arcs of G are mapped to half-lines meeting at $g(i) = 0$, and G is mapped onto a sector S of angle $3\pi/2$ (see 10.7 and 10.14). By conformality $g(0) = -1$ must lie on the bisector of the complementary sector, so S is as shown in Fig. 10.10. Note how we have avoided having to compute the angles subtended by the boundary arcs of G.

Stage 2 Working from the opposite end, we can realize $\hat{G} = \{w : |w| > 1\}$ as the image under the conformal map

$$h : \tau \mapsto \frac{\tau+1}{\tau-1} = w$$

of the right half-plane $\hat{S} = \{\tau : |\tau - 1| < |\tau + 1|\}$.

Stage 3 To transform S onto $\hat{S}$ we seek to multiply angles at 0 by a factor of $\frac{2}{3}$. Formally $\zeta \mapsto \tau = \zeta^{\frac{2}{3}}$ provides the map we want, but

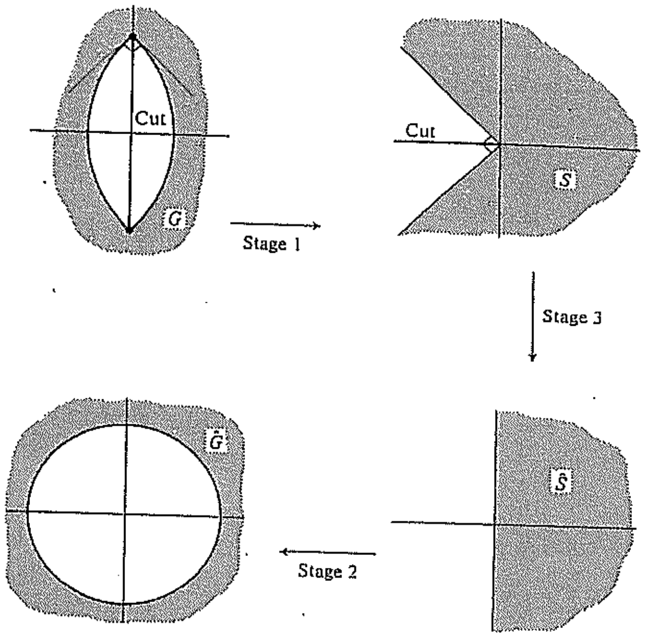

Fig. 10.10

we have to take care because this is a multifunction. We start with the z-plane cut along $[-i, i]$. In this cut plane there exists a holomorphic branch k of $[[(z-i)/(z+i)]^{\frac{2}{3}}]$. The map we finally require is $f = h \circ k$. This sends z to w where $[(z-i)/(z+i)]^2 = \tau^3$ and $(\tau+1)/(\tau-1) = w$. Hence $f(z) = w$, where

$$\left(\frac{z-i}{z+i}\right)^2 = \left(\frac{w+1}{w-1}\right)^3. \qquad \square$$

10.21 Remarks

Because of the way they act on circlines, Möbius transformations, powers, and exponentials are of most use for mapping regions whose boundaries are made up of circular arcs, lines, and line segments. However their scope is, even so, limited. For example, to handle regions with polygonal boundaries, one must introduce the *Schwarz-Christoffel transformation*, which is awkwardly defined by an integral:

$$z \mapsto \int_0^z (\zeta - z_1)^{-k_1} (\zeta - z_2)^{-k_2} \ldots (\zeta - z_n)^{-k_n} \, d\zeta.$$