If A and B are two classes of coterminal sequences, we define $A \pm B$ as the class of all sequences of the form $\{u_n \pm v_n\}$, AB as the class of all sequences of the form $\{u_n v_n\}$, and A/B as the class of all sequences of the form $\{u_n/v_n\}$, where $\{u_n\} \in A$, $\{v_n\} \in B$ but are otherwise arbitrary.[2] Since $\{u_n\} \in A$, $\{v_n\} \in B$, we have

$$\lim_{n\to\infty} u_n = a, \quad \lim_{n\to\infty} v_n = b$$

and hence

$$\lim_{n\to\infty} (u_n \pm v_n) = a \pm b, \quad \lim_{n\to\infty} u_n v_n = ab, \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{a}{b}. \tag{5.2}$$

It follows that $A \pm B$, AB and A/B are again classes of coterminal sequences. Moreover (5.2) implies that

$$a \pm b \leftrightarrow A \pm B, \quad ab \leftrightarrow AB, \quad \frac{a}{b} \leftrightarrow \frac{A}{B},$$

i.e., the one-to-one correspondence (5.1) is an *isomorphism*[3] between the field of complex numbers and the field of coterminal sequences. In this sense, we are justified in identifying a and A, b and B, etc., and then every class of coterminal sequences is called a *proper complex number*. In particular, the class A can be represented geometrically by the same point in the complex plane as that representing the complex number a. With this interpretation, a convergent sequence $\{u_n\}$ belongs to the class $A = a$ if and only if the unique limit point of the set of points representing the complex numbers $u_1, u_2, \ldots, u_n, \ldots$ coincides with a.

We now adjoin a single *improper complex number* to the set of all proper complex numbers (i.e., the set of all classes of coterminal sequences). This improper complex number, which we denote by ∞, is the class of all sequences $\{u_n\}$ with the property that given any $\rho > 0$, there exists an integer $n_0 > 0$ (depending on ρ and $\{u_n\}$) such that $|u_n| > \rho$ whenever $n > n_0$. If a sequence $\{u_n\}$ belongs to the class ∞, we say that $\{u_n\}$ *converges to infinity*, and we write $u_n \to \infty$ as $n \to \infty$ or

$$\lim_{n\to\infty} u_n = \infty.$$

The union of the set of all proper complex numbers and the improper complex number ∞ is called the extended complex number system. Algebraic operations are defined for the extended complex number system in exactly the same way as for the set of proper complex numbers. However, as the following examples show, in some cases an algebraic expression involving the class ∞ does not lead to a class of coterminal sequences, and hence is meaningless:

[2] Whenever we write u_n/v_n or a/b, it is assumed that $v_n \neq 0$ or $b \neq 0$.
[3] See e.g., G. Birkhoff and S. MacLane, *op. cit.*, p. 42.

1. If a is a proper complex number, then $a + \infty = \infty + a = \infty$, but $\infty + \infty$ is meaningless, as shown by the two sequences

$$1, 3, 3, 5, 5, 7, 7, 9, \ldots \tag{5.3}$$

and

$$-1, -2, -3, -4, -5, -6, -7, -8, \ldots \tag{5.4}$$

belonging to the class ∞. In fact, adding (5.3) and (5.4) term by term, we obtain the sequence

$$0, 1, 0, 1, 0, 1, 0, 1, \ldots,$$

which obviously does not belong to any class of coterminal sequences (since it has no limit).

2. Similarly, $a - \infty = \infty - a = \infty$ if a is a complex number, but $\infty - \infty$ is meaningless.
3. If $a \neq 0$ is a proper complex number, then $a \cdot \infty = \infty \cdot a = \infty$ and moreover $\infty \cdot \infty = \infty$, but $0 \cdot \infty$ is meaningless.
4. If a is any proper complex number, then $a/\infty = 0$ and $\infty/a = \infty$, but ∞/∞ is meaningless.

Remark. This approach also allows us to divide by zero. In fact, $a/0 = \infty$ if $a \neq 0$ is a proper complex number, but $0/0$ is meaningless.

21. Stereographic Projection. Sets of Points on the Riemann Sphere

In order to represent the extended complex number system geometrically, it is convenient to use the following construction, due to Riemann (1826–1866). Consider a sphere Σ of unit radius and center O, and let Π be a plane passing through O (see Figure 5.1). Introducing a rectangular coordinate system in the plane Π, with origin at O, we can represent any proper complex number $z = x + iy$ by a point (x, y) in the plane Π. To associate a point on the sphere Σ with a given point $P \in \Pi$, we first draw the diameter NS of the sphere which is perpendicular to Π and intersects Π at O. Then we draw the line segment joining one end of this diameter, say N, to the point P. The line segment NP (or its prolongation) intersects the sphere Σ in some point P^* different from N. It is clear that this construction establishes a one-to-one correspondence between the points of the sphere Σ (except for the point N itself) and the points of the plane Π. This mapping of the sphere into the plane (or of the plane into the sphere) is called *stereographic projection*, and the sphere Σ is called the *Riemann sphere*. If the $P \in \Pi$ represents the complex number z, we also regard the point $P^* \in \Sigma$ as representing z.

To be as descriptive as possible, we use geographic terminology. Thus, the circle in which the sphere Σ intersects the plane is called the *equator*, the points N and S are called the *north pole* and *south pole*, respectively, the great circles going through N and S are called *meridians*, and in particular, the meridian lying in the plane NOx is called the *prime* (or *initial*) *meridian*. Then it is easy to see that the points of the plane Π lying inside the unit circle $|z| = 1$ (which coincides with the equator) are mapped into points of the *southern hemisphere* (containing S), while the points of Π lying outside the unit circle are mapped into points of the *northern hemisphere* (containing N). Similarly, the upper half-plane $y > 0$ is mapped into the *eastern hemisphere* (which is intersected by the positive y-axis), while the lower half-plane $y < 0$ is mapped into the *western hemisphere*, and so on.

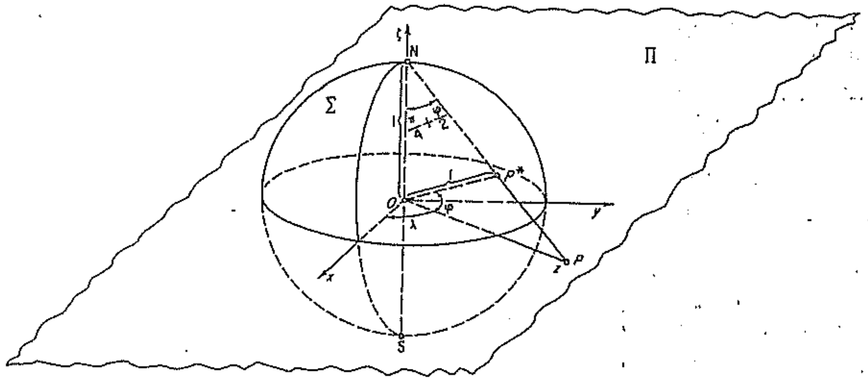

FIGURE 5.1

We now introduce spherical (or geographic) coordinates on Σ, i.e., the *latitude* φ, measured from the equator and ranging from 0 to $\pi/2$ in the northern hemisphere and from 0 to $-\pi/2$ in the southern hemisphere, and the *longitude* λ, measured from the prime meridian (more exactly, from the point of intersection of the prime meridian with the positive x-axis) and ranging from 0 to π (including π) in the eastern hemisphere and from 0 to $-\pi$ (excluding $-\pi$) in the western hemisphere. As shown by Figure 5.1, under stereographic projection we have

$$\arg z = \lambda, \qquad |z| = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

Therefore, the point of the sphere with coordinates λ and φ is the image of the complex number

$$z = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)(\cos\lambda + i\sin\lambda). \tag{5.5}$$

Conversely, the image on the Riemann sphere of the complex number $z \neq 0$ has spherical coordinates

$$\lambda = \arg z, \qquad \varphi = 2\arctan|z| - \frac{\pi}{2}.$$

Next we derive the formulas relating the rectangular coordinates (ξ, η, ζ) of the point P^* on the Riemann sphere and the rectangular coordinates $(x, y, 0)$ of the point $P \in \Pi$ which corresponds to P^* under stereographic projection. Since

$$\xi = \cos\varphi\cos\lambda, \quad \eta = \cos\varphi\sin\lambda, \quad \zeta = \sin\lambda,$$

and

$$\cos\varphi = \sin\left(\frac{\pi}{2} + \varphi\right) = \frac{2\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{1 + \tan^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)} = \frac{2|z|}{1 + |z|^2},$$

$$\sin\varphi = -\cos\left(\frac{\pi}{2} + \varphi\right) = -\frac{1 - \tan^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{1 + \tan^2\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)} = \frac{|z|^2 - 1}{1 + |z|^2},$$

it follows that

$$\begin{aligned} \xi &= \cos\varphi\cos\lambda = \frac{|z|\cos\lambda}{1 + |z|^2} = \frac{2x}{x^2 + y^2 + 1}, \\ \eta &= \cos\varphi\sin\lambda = \frac{|z|\sin\lambda}{1 + |z|^2} = \frac{2y}{x^2 + y^2 + 1}, \\ \zeta &= \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}. \end{aligned} \tag{5.6}$$

Moreover

$$z = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)(\cos\lambda + i\sin\lambda) = \frac{1 + \tan\frac{\varphi}{2}}{1 - \tan\frac{\varphi}{2}}(\cos\lambda + i\sin\lambda)$$

$$= \frac{\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}}(\cos\lambda + i\sin\lambda) = \frac{\cos\varphi}{1 - \sin\varphi}(\cos\lambda + i\sin\lambda) = \frac{\xi + i\eta}{1 - \zeta},$$

and hence

$$x = \frac{\xi}{1 - \zeta}, \qquad y = \frac{\eta}{1 - \zeta}. \tag{5.7}$$

Using these formulas, we can prove the *circle-preserving property* of stereographic projection, i.e., we can show that stereographic projection maps circles on the Riemann sphere into circles or straight lines in the plane Π (the latter correspond to circles on Σ which go through the north pole N). To verify this, we first note that any circle on Σ is the intersection of Σ with some plane

$$A\xi + B\eta + C\zeta + D = 0. \tag{5.8}$$

It follows from (5.8) and (5.6) that the corresponding points of the plane satisfy the equation

$$Ax + By + (C + D)(x^2 + y^2) + (D - C) = 0,$$

which is the equation of a circle if $C + D \neq 0$, and the equation of a straight line if $C + D = 0$. But $C + D = 0$ is the result obtained when we substitute the coordinates (0, 0, 1) of the north pole into (5.8). Thus we obtain a circle or a straight line in the plane Π, depending on whether or not the circle on the sphere goes through the north pole N.

In particular, it should be noted that if the plane (5.8) is parallel to the equatorial plane $\zeta = 0$, i.e., if the circle on the sphere is a parallel of latitude, then $A = B = 0$, and in the plane Π we obtain a circle

$$(C + D)(x^2 + y^2) = C - D$$

with its center at the origin of coordinates. On the other hand, if the plane (5.8) goes through the axis NS, i.e., if the circle on the sphere is one of the meridians, then $C = D = 0$, and in the plane Π we obtain a straight line

$$Ax + By = 0$$

going through the origin of coordinates. Of course, all these results are apparent almost at once from the geometric meaning of stereographic projection.

Next we discuss sets of points on the Riemann sphere. Every circle γ on the sphere Σ which does not go through a given point $P^* \in \Sigma$ divides Σ into two parts, such that one part contains P^* and the other does not. The part of Σ containing P^* (and not including γ) will be called a *neighborhood* of P^* (cf. footnote 1, p. 27). Once the concept of the neighborhood of a point has been introduced, we can immediately extend the concepts of a limit point, a closed set, an open set, a continuous curve, a domain, etc., to sets of points on the surface Σ.

Example 1. A point $P^* \in \Sigma$ is a limit point of a given set $E \subset \Sigma$ if and only if any neighborhood of P^* contains infinitely many (distinct) points of E.

Example 2. Every triple of real functions

$$\xi = \varphi(t), \quad \eta = \psi(t), \quad \zeta = \chi(t)$$

of the real parameter t, defined and continuous on a closed interval $a \leqslant t \leqslant b$, and satisfying the condition

$$[\varphi(t)]^2 + [\psi(t)]^2 + [\chi(t)]^2 = 1,$$

defines a continuous curve on Σ.

Example 3. Let $\{a_n\}$ be a sequence of complex numbers converging to a (proper) complex number a. Let a_n $(n = 1, 2, \ldots)$ be represented by the point P_n in the plane Π and by the point P_n^* on the sphere Σ, and let a be represented by the point $P \in \Pi$ and by the point $P^* \in \Sigma$. Then P^* is the unique limit point of the sequence of points $\{P_n^*\}$, just as P is the unique limit point of the sequence of points $\{P_n\}$.

Now consider a sequence $\{a_n\}$ belonging to the improper class ∞ (see p. 78). Then, given any $\rho > 0$, there exists an integer $n > 0$ such that the points P_n representing the numbers a_n in the complex plane lie outside the circle $|z| = \rho$ if $n > n_0$. The corresponding points P_n^* of the Riemann sphere lie in the neighborhood of the north pole N bounded by the parallel of latitude whose projection onto the plane is the circle $|z| = \rho$. Therefore, the north pole is the unique limit point of every sequence belonging to the improper class. Conversely, every sequence of points on the sphere for which the north pole is the unique limit point is a sequence belonging to the improper class. This follows at once from the formula

$$|z| = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$$

derived above, according to which the relations

$$\lim_{n \to \infty} |z_n| = +\infty$$

and

$$\lim_{n \to \infty} \varphi_n = \frac{\pi}{2}$$

are equivalent.[4] In keeping with this fact, *we shall henceforth regard the north pole N of the Riemann sphere as the geometric image of the improper complex number* ∞.

22. The Extended Complex Plane. The Point at Infinity

Let Σ denote the Riemann sphere, let $\Sigma - \{N\}$ denote Σ with the single point N (the north pole) deleted, and let Π be the ordinary or *finite* (complex) plane. Then, as we have just seen, stereographic projection is a one-to-one

[4] Note the distinction between the real number $+\infty$ and the complex number ∞. However, we usually write ∞ instead of $+\infty$ whenever $+\infty$ is a value of a real quantity which is inherently nonnegative, or when the context precludes any possibility of confusion