distinguish these branches, we can fix one of the two values of $f(z)$ at some point of the domain G, e.g., at the point ∞. According to (11.13), z becomes infinite either for $w = 0$ or $w = \infty$. Therefore the branch $f_1(z)$ of (11.12) satisfying the condition $f_1(\infty) = 0$ maps G onto $I(\gamma)$, while the branch $f_2(z)$ satisfying the condition $f_2(\infty) = \infty$ maps G onto $E(\gamma)$. Instead of G, we might have chosen the domain G' whose boundary consists of the infinite segment of the real axis joining the points -1 and 1, or the domains G_U and G_L, whose boundaries are the upper and lower unit semicircles.

56. The Logarithm

The inverse of the function

$$z = e^w = e^u(\cos v + i \sin v)$$

is defined for any value of z different from 0 and ∞, and is represented by the formula

$$w = \ln |z| + i \operatorname{Arg} z$$

[cf. (9.25)]. This function, which is obviously multiple-valued (in fact, *infinite-valued*), is called the *logarithm* and is denoted by $\operatorname{Ln} z$, i.e.,

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z, \qquad (11.14)$$

by definition. The value

$$\ln |z| + i \arg z$$

of the logarithm is called the *principal value*, and is denoted by $\ln z$. Then (11.14) can be written in the form

$$\operatorname{Ln} z = \ln z + 2k\pi i \qquad (k = 0, \pm 1, \pm 2, \ldots). \qquad (11.15)$$

It follows that *every complex number different from* 0 *and* ∞ *has infinitely many logarithms (i.e., values of the function* $\operatorname{Ln} z$*), and any two of these logarithms differ by an integral multiple of* $2\pi i$.

If z is a positive real number, the principal value of the logarithm is just $\ln |z|$, which is exactly what is meant by the logarithm in elementary mathematics; for example,

$$\ln 1 = 0, \quad \ln e = 1, \quad \ln 2 = 0.69315\ldots$$

For negative real numbers and for imaginary numbers, the principal value of the logarithm is an imaginary number

$$\ln |z| + i \arg z \qquad (\arg z \neq 0, \quad -\pi < \arg z \leqslant \pi)$$

and all the other values of the logarithm are also imaginary numbers,

calculated by using (11.15); for example,

$$\operatorname{Ln}(-1) = (2k+1)\pi i,$$
$$\operatorname{Ln}(-2) = 0.69315\ldots + (2k+1)\pi i,$$
$$\operatorname{Ln}(1-i) = \ln\sqrt{2} - \frac{\pi i}{4} + 2k\pi i = 0.34657\ldots + (8k-1)\frac{\pi i}{4}.$$

The familiar rules for finding logarithms of products and quotients remain valid for the multiple-valued logarithms of complex numbers, since

$$\begin{aligned}\operatorname{Ln}(z_1 z_2) &= \ln |z_1 z_2| + i \operatorname{Arg}(z_1 z_2)\\ &= \ln |z_1| + \ln |z_2| + i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2)\\ &= \operatorname{Ln} z_1 + \operatorname{Ln} z_2\end{aligned} \qquad (11.16)$$

and

$$\begin{aligned}\operatorname{Ln}\frac{z_1}{z_2} &= \ln\frac{z_1}{z_2} + i \operatorname{Arg}\frac{z_1}{z_2}\\ &= \ln |z_1| - \ln |z_2| + i(\operatorname{Arg} z_1 - \operatorname{Arg} z_2)\\ &= \operatorname{Ln} z_1 - \operatorname{Ln} z_2,\end{aligned} \qquad (11.17)$$

where z_1 and z_2 are arbitrary nonzero complex numbers. In (11.16) and (11.17), both the left and right-hand sides (for fixed z_1 and z_2) represent infinite sets of complex numbers, and the equalities have to be understood in the sense that these two sets are equal, i.e., have the same members. Failure to remember this fact can lead to paradoxical results. For example, in a sophism constructed by John Bernoulli, it is claimed that $\operatorname{Ln}(-z) = \operatorname{Ln} z$ for arbitrary $z \neq 0$, and the following chain of equalities is adduced as "proof":

1. $\operatorname{Ln}[(-z)^2] = \operatorname{Ln}(z^2)$;
2. $\operatorname{Ln}(-z) + \operatorname{Ln}(-z) = \operatorname{Ln} z + \operatorname{Ln} z$;
3. $2 \operatorname{Ln}(-z) = 2 \operatorname{Ln} z$;
4. $\operatorname{Ln}(-z) = \operatorname{Ln} z$.

However, the conclusion that $\operatorname{Ln}(-z) = \operatorname{Ln} z$ is false, since

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z = \ln |z| + i \arg z + 2k\pi i,$$
$$\operatorname{Ln}(-z) = \ln |-z| + i \operatorname{Arg}(-z) = \ln |z| + i \arg z + (2k+1)\pi i,$$

and obviously none of the numbers representing the values of $\operatorname{Ln} z$ is the same as any of the numbers representing the values of $\operatorname{Ln}(-z)$. The fallacy in the "proof" occurs in going from equality 2 to equality 3. The first of these relations is based on formula (11.16), and is of course true. However, the sum $\operatorname{Ln}(-z) + \operatorname{Ln}(-z)$ cannot be replaced by $2 \operatorname{Ln}(-z)$, since the sum in question is obtained from the set of numbers $\operatorname{Ln}(-z)$ by adding each of these numbers to itself and to *all the other* numbers of the set $\operatorname{Ln}(-z)$, whereas the set $2 \operatorname{Ln}(-z)$ is obtained by simply doubling all the numbers $\operatorname{Ln}(-z)$, i.e., by adding each such number *to itself only.*[7]

[7] The following simple example may help clarify the situation: Let A be the set consisting of the two numbers 0 and 1. Then $A + A$ is the set consisting of the three numbers $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 2$, whereas the set $2A$ consists only of two numbers $2 \cdot 0 = 0$ and $2 \cdot 1 = 2$.

Therefore

$$\text{Ln}\,(-z) + \text{Ln}\,(-z) \neq 2\,\text{Ln}\,(-z),$$

and by the same token,

$$\text{Ln}\,z + \text{Ln}\,z \neq 2\,\text{Ln}\,z.$$

Setting $z_1 = z_2 = z \neq 0$ in (11.17), we obtain the relation

$$\text{Ln}\,1 = \text{Ln}\,z - \text{Ln}\,z, \tag{11.18}$$

which is a correct formula. However, the right-hand side of (11.18) cannot be replaced by 0, since here we are talking about the set of all possible differences between values of the logarithm of the same number. This set consists of all possible multiples of $2\pi i$, so that to be perfectly explicit, we should write (11.18) as

$$\text{Ln}\,1 = 2k\pi i \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

We now study the single-valued branches of the logarithm. We begin by finding domains of univalence for the exponential function $z = e^w$, which is the inverse of the logarithm $w = \text{Ln}\,z$. All the numbers w for which e^w takes any given value z are given by the formula

$$w = \ln|z| + i\,\text{Arg}\,z$$

[cf. (9.95)], i.e., all the numbers w can be obtained by shifting any one of them by $2k\pi i$, where $k = 0, \pm 1, \pm 2, \ldots$. Therefore a domain of univalence for $z = e^w$ cannot contain any pair of points such that one point can be obtained from the other by a shift of this kind. The simplest way to satisfy this requirement is to start with an open rectilinear strip

$$\mathscr{G}_0 : v_0 < v < v_0 + 2\pi$$

of width 2π parallel to the real axis in the w-plane. Then, subjecting $\mathscr{G}_0$ to all possible shifts of the form $2k\pi i$, we obtain an infinite family of domains of univalence:

$$\mathscr{G}_k : v_0 + 2k\pi < v < v_0 + 2(k+1)\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Obviously, every point of the w-plane is either an interior point of a domain $\mathscr{G}_k$ or a boundary point of two domains $\mathscr{G}_k$ and $\mathscr{G}_{k+1}$ [see Figure 11.5(b)].

The image (under $z = e^w$) of each of the strips $\mathscr{G}_k$ is the same domain G in the z-plane, i.e., the interior of an angle of 2π radians with its vertex at the origin. The boundary of G is a single ray of inclination v_0 emanating from the origin [see Figure 11.5(a)]. We can define infinitely many different single-valued branches of Ln z on the domain G by specifying that the kth branch $\text{Ln}_k\,z$ $(k = 0, \pm 1, \pm 2, \ldots)$ have $\mathscr{G}_k$ as its range. Each of these branches is uniquely characterized by the value w_0 which it assigns to any given

point $z_0 \in G$, since one and only one of the domains $\mathscr{G}_k$ contains the point w_0. Expressed somewhat differently,

$$\text{Ln}_k\,z = \ln|z| + i\,\text{Arg}_k\,z,$$

where $\text{Arg}_k\,z$ is the value of the argument satisfying the inequality

$$v_0 + 2k\pi < \text{Arg}_k\,z < v_0 + 2(k+1)\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Moreover, since $w = \text{Ln}_k\,z$ is a one-to-one continuous mapping of G onto $\mathscr{G}_k$, and since $z = e^w$ has a nonzero derivative e^w on $\mathscr{G}_k$, the branches $\text{Ln}_k\,z$ all have nonzero derivatives on G, i.e.,

$$\frac{d}{dz}\,\text{Ln}_k\,z = \frac{1}{e^w} = \frac{1}{z} \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

(cf. Rule 5, p. 109).

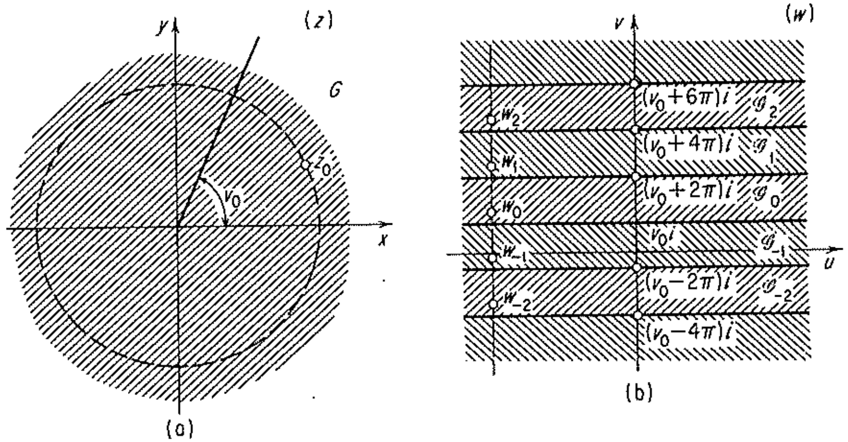

FIGURE 11.5

The points 0 and ∞ are both branch points of the function Ln z. To see this, suppose that at a point $z_0 \in G$ we choose a value of Ln z corresponding to the branch $\text{Ln}_k\,z$ and represented by the point

$$w_k = \text{Ln}_k\,z_0 = \ln|z_0| + i\,\text{Arg}_k\,z_0,$$

belonging to the strip $\mathscr{G}_k$. Then, as z moves continuously around the circle $|z| = |z_0|$ in the counterclockwise direction, starting from the point z_0, the value of

$$w = \ln|z| + i\,\text{Arg}\,z \tag{11.19}$$

changes continuously, and when z returns to its original value z_0, (11.19) goes into the value

$$w_{k+1} = \ln |z_0| + i \operatorname{Arg}_k z_0 + 2\pi i = \ln |z_0| + i \operatorname{Arg}_{k+1} z_0 = \operatorname{Ln}_{k+1} z_0.$$

Thus, since the point $z_0 \in G$ is arbitrary, one circuit around the origin $z = 0$ (or about the point at infinity $z = \infty$) causes the branch $\operatorname{Ln}_k z$ to change continuously into the branch $\operatorname{Ln}_{k+1} z$.[8] Obviously, as we make additional circuits around the origin in the counterclockwise direction, the branch $\operatorname{Ln}_k z$ undergoes the infinite sequence of transformations

$$\operatorname{Ln}_k z \to \operatorname{Ln}_{k+1} z, \quad \operatorname{Ln}_{k+1} z \to \operatorname{Ln}_{k+2} z, \quad \operatorname{Ln}_{k+2} z \to \operatorname{Ln}_{k+3} z, \ldots,$$

so that $\operatorname{Ln}_k z$ is never carried back into itself. For this reason, the points 0 and ∞ are called *branch points of infinite order* or *logarithmic branch points.*

Remark. Single-valued branches of the function $\operatorname{Ln} z$ can be defined on domains more general than G. Let γ be any Jordan curve joining the points $z = 0$ and $z = \infty$. Then the curve γ has infinitely many images Γ_k ($k = 0, \pm 1, \pm 2, \ldots$) under the mapping $w = \operatorname{Ln} z$. The curves Γ_k, which are all Jordan curves, divide the w-plane into infinitely many open curvilinear strips $\mathscr{D}_k$, where the boundary of $\mathscr{D}_k$ consists of the pair of curves Γ_k and Γ_{k+1}, and $\mathscr{D}_k$ does not contain any of the other curves Γ_k. Thus we can define a countable family of single-valued branches of $\operatorname{Ln} z$ on the domain D with boundary γ by specifying that the kth branch $(\operatorname{Ln} z)_k$, where $k = 0, \pm 1, \pm 2, \ldots$, have $\mathscr{D}_k$ as its range.[9] Clearly,

$$(\operatorname{Ln} z)_k = (\operatorname{Ln} z)_l + 2\pi(k - l)i,$$

and, as before, we find that

$$\frac{d}{dz} (\operatorname{Ln} z)_k = \frac{1}{z}. \tag{11.20}$$

Since (11.20) is independent of how the branches of $\operatorname{Ln} z$ are defined, we can simply write

$$\frac{d}{dz} \operatorname{Ln} z = \frac{1}{z},$$

where by the left-hand side we mean any single-valued branch of $\operatorname{Ln} z$, defined on a domain containing the given point z.

57. The Function z^α. Exponentials and Logarithms to an Arbitrary Base

We begin by defining the expression a^α, where $a \neq 0$ and α are arbitrary complex numbers. We first assume that α is real, and examine in turn the cases where α is an integer, a rational number and an irrational number.

[8] Similarly, one circuit around the origin in the opposite (i.e., clockwise) direction causes the branch $\operatorname{Ln}_k z$ to change continuously into the branch $\operatorname{Ln}_{k-1} z$.

[9] Note the distinction between $\operatorname{Ln}_k z$ and $(\operatorname{Ln} z)_k$.

Case 1. If $\alpha = n$ is an integer, then

$$a^\alpha = a^n = |a|^n [\cos (n \operatorname{Arg} a) + i \sin (n \operatorname{Arg} a)], \tag{11.21}$$

and a^α *has just one value.*

Case 2. If $\alpha = r$ is a rational number, then $r = m/n$ where m and $n > 0$ are relatively prime integers. As we already know [cf. formula (2.15)], in this case,

$$\begin{aligned} a^\alpha = a^{m/n} &= |a|^{m/n} \left[\cos \left(\frac{m}{n} \operatorname{Arg} a\right) + i \sin \left(\frac{m}{n} \operatorname{Arg} a\right)\right] \\ &= |a|^r [\cos (r \operatorname{Arg} a) + i \sin (r \operatorname{Arg} a)], \end{aligned}$$

and a^α *has* n *different values.*

Case 3. If $\alpha = \rho$ is an irrational number, we define a^ρ by continuity, i.e., as the limit

$$\lim_{n\to\infty} a^{r_n} = \lim_{n\to\infty} |a|^{r_n} [\cos (r_n \operatorname{Arg} a) + i \sin (r_n \operatorname{Arg} a)], \tag{11.23}$$

where $\{r_n\}$ is an arbitrary sequence of irrational numbers converging to ρ. In taking the limit (11.23), we hold $\operatorname{Arg} a$ fixed. Then

$$\lim_{n\to\infty} r_n \operatorname{Arg} a = \rho \operatorname{Arg} a,$$

and (11.23) implies (cf. p. 34)

$$a^\rho = |a|^\rho [\cos (\rho \operatorname{Arg} a) + i \sin (\rho \operatorname{Arg} a)]. \tag{11.24}$$

To obtain all the values of a^ρ, we now let $\operatorname{Arg} a$ take all its values. Since two values of $\rho \operatorname{Arg} a$ differ by a number of the form $2k\rho\pi$, where k is a nonzero integer, and since $k\rho$ can never be an integer, it follows that different values of $\operatorname{Arg} a$ give rise to different values of a^ρ. Thus, in this case, a^α *has infinitely many different values.*

Remark. It should be noted that formulas (11.21), (11.22) and (11.24) are all special cases of the formula

$$a^\alpha = |a|^\alpha [\cos (\alpha \operatorname{Arg} a) + i \sin (\alpha \operatorname{Arg} a)],$$

which can be written in the form

$$\begin{aligned} a^\alpha &= \exp (\alpha \ln |a|)[\cos (\alpha \operatorname{Arg} a) + i \sin (\alpha \operatorname{Arg} a)] \\ &= \exp (\alpha \ln |a| + i\alpha \operatorname{Arg} a) = \exp (\alpha \operatorname{Ln} a). \end{aligned} \tag{11.25}$$

For the time being, we make a distinction between $\exp z$ and e^z, with only the former being used to denote the exponential function defined in Sec. 38 (see Example 2 below).

Now let α be an arbitrary complex number. Observing that in this case the right-hand side of (11.25) still has meaning, we write

$$a^\alpha = \exp (\alpha \operatorname{Ln} a),$$

by definition. If α is imaginary, all the values of a^α corresponding to different values of Ln a (or equivalently, to different values of Arg a) are also different, since two distinct values of α Ln a differ by a number of the form $2\pi i\alpha$, which cannot be an integral multiple of $2\pi i$ if α is imaginary. It follows from (11.25) and the addition theorem for the exponential that

$$a^{\alpha_1}a^{\alpha_2} = \exp(\alpha_1 \operatorname{Ln} a)\exp(\alpha_2 \operatorname{Ln} a) = \exp[(\alpha_1 + \alpha_2)\operatorname{Ln} a] = a^{\alpha_1+\alpha_2}. \tag{11.26}$$

Moreover, we have the following rule for raising a power to a power:

$$\begin{aligned}(a^\alpha)^\beta &= [\exp(\alpha \operatorname{Ln} a)]^\beta = \exp\{\beta \operatorname{Ln}[\exp(\alpha \operatorname{Ln} a)]\} \\ &= \exp(\beta\alpha \operatorname{Ln} a) = a^{\alpha\beta}.\end{aligned} \tag{11.27}$$

Both (11.26) and (11.27) are the same as the corresponding rules for real numbers.

Example 1.

$$\begin{aligned}1^{\sqrt{2}} &= \exp(\sqrt{2}\ln 1) = \exp(2k\pi\sqrt{2}i) \\ &= \cos(2k\pi\sqrt{2}) + i\sin(2k\pi\sqrt{2}) \qquad (k = 0, \pm 1, \pm 2, \ldots).\end{aligned}$$

Example 2.

$$\begin{aligned}e^z &= \exp(z \operatorname{Ln} e) = \exp[z(1 + 2k\pi i)] \\ &= \exp z \exp(2k\pi iz) \qquad (k = 0, \pm 1, \pm 2, \ldots).\end{aligned} \tag{11.28}$$

It follows from (11.28) that only one of the values of the power e^z coincides with exp z. In fact, the other values are

$$\exp z \exp(2\pi iz), \quad \exp z \exp(-2\pi iz), \ldots$$

In particular, only one of the values of e^x (where x is a real number) coincides with the positive real number exp x, and in this case, the other values are

$$\exp x \exp(2\pi ix), \quad \exp x \exp(-2\pi ix), \ldots \tag{11.29}$$

[If x is rational, there are only a finite number of different values (11.29), but if x is irrational, there are infinitely many different values.] Despite this, *we shall continue to denote the exponential function by* e^z, *as well as by* exp z. This use of the "multiple-valued symbol" e^z to denote a single number is analogous to the conventional use of the symbol $\sqrt[n]{a}$ (where a is a positive real number) to denote the unique positive value of the nth root of a (cf. footnote 6, p. 18).

Example 3.

$$i^i = \exp(i \operatorname{Ln} i) = \exp\left[i\left(\frac{\pi i}{2} - 2k\pi i\right)\right] = e^{(4k-1)\pi/2} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Thus all the infinitely many values of i^i are positive real numbers, which can be arbitrarily small or arbitrarily large.

We are now in a position to study the functions z^α and a^z, where α and $a \neq 0$ are arbitrary complex numbers. First we consider the function

$$f(z) = z^\alpha$$

(which in general is defined only for $z \neq 0$), and examine in turn the cases where α is an integer, a rational number and an irrational or imaginary number.

Case 1. If $\alpha = n$ is an integer, then

$$f(z) = z^\alpha = z^n$$

is a particularly simple rational function. In this case, $f(z)$ is defined for $z = 0$, where it has a zero (if $n > 0$) or a pole (if $n < 0$).

Case 2. If $\alpha = r$ is a rational number, then $r = m/n$ where m and $n > 0$ are relatively prime integers, and hence

$$f(z) = z^{m/n} = \sqrt[n]{z^m}.$$

In this case, $f(z)$ is an n-valued function, with branch points $z = 0$ and $z = \infty$ of order $n - 1$. Let γ be any Jordan curve joining 0 and ∞, and let G be the domain with boundary γ. Then we can define n single-valued differentiable branches of $f(z)$ on G, which change continuously into each other as we go around any closed Jordan curve whose interior contains the origin.

Case 3. If α is not a rational number, i.e., if α is an irrational real number or an imaginary number, then $f(z) = z^\alpha$ is an infinite-valued function, all of whose values are given by the formula

$$z^\alpha = \exp(\alpha \operatorname{Ln} z).$$

In this case, $z = 0$ and $z = \infty$ are branch points (as in Case 2), but now they are of infinite order. In fact, if we go around the point $z = 0$ (say, in the positive direction), Arg z varies continuously and increases by 2π. Therefore the value of α Ln z increases by $2\pi i\alpha$, i.e., $f(z)$ is multiplied by factor $e^{2\pi i\alpha} \neq 1$.

Next we consider the *exponential to the base* a, i.e., the function

$$a^z = \exp(z \operatorname{Ln} a),$$

defined for any finite z and any $a \neq 0$. To obtain a definite single-valued branch of $f(z)$, it is sufficient to fix one of the values of Ln a, say

$$\operatorname{Ln} a = b = \ln a + 2k_0\pi i. \tag{11.30}$$

After this has been done, we obtain a single-valued, everywhere differentiable function exp (bz). Taking all possible values of Ln a, we obtain all possible single-valued branches of the function a^z. Since two values of Ln a differ by a term of the form $2k\pi i$, where k is an integer, two branches of the function a^z differ by a factor of the form $\exp(2k\pi iz)$. This factor is also a single-valued, everywhere differentiable function, which takes the value 1 only when z is a rational number of the form m/k (where m is an arbitrary integer).

Remark. The branches of the multiple-valued function a^z differ in an essential way from those of the multiple-valued functions considered previously. In all the cases considered so far, we can find points of the extended plane, called *branch points*, with the property that by going around each branch point along suitable closed Jordan curves we can carry any single-valued branch into any other. However, in the case of the function a^z, this is impossible, since every branch is a single-valued continuous function defined on the whole finite plane (and not on some domain whose boundary consists of certain curves joining the branch points). In fact, after making a circuit around any (finite) closed Jordan curve whatsoever, we return to the original complex number z (perhaps with a different value of the argument) and hence to the same value of the function $\exp(bz)$, where b is a fixed value of Ln a. Thus the multiple-valued function a^z has no branch points at all, and its single-valued branches cannot be carried continuously into each other by making circuits around closed Jordan curves. In other words, the different branches can be regarded as self-contained, independent, single-valued, everywhere differentiable functions,[10] i.e., as nothing more or less than the infinite set of entire functions

$$\exp(z \ln a), \quad \exp[z(\ln a + 2\pi i)], \quad \exp[z(\ln a - 2\pi i)], \ldots$$

The fact that all these different entire functions can be represented as branches of a single infinite-valued function a^z is no more surprising than the fact that $\sin z$ and $-\sin z$ can be regarded as branches of the double-valued function

$$\sqrt{1 - \cos^2 z}, \tag{11.31}$$

or that $\cosh z$ and $\sinh z$ can be regarded as branches of the double-valued function

$$\tfrac{1}{2}[\exp z + \sqrt{\exp(-2z)}]. \tag{11.32}$$

[It should be noted that just like the function a^z, the functions (11.31) and (11.32) have no branch points.]

Finally we consider the *logarithm to the base a*, denoted by $\text{Log}_a z$ and defined as the inverse of the function

$$z = a^w = \exp(w \,\text{Ln}\, a). \tag{11.33}$$

[10] The same situation has already been encountered in Example 2 above, in connection with the function e^z (where $a = e$ and $\ln a = 1$). However, in the case of e^z, we agreed to interpret e^z as the particular single-valued branch $\exp z$, i.e., as the branch which takes real values when z is real (see p. 136). We shall make the same choice whenever a is a positive real number.

If we again write (11.30), thereby choosing a branch of a^w, (11.33) becomes

$$z = \exp(bw),$$

and hence

$$w = \text{Log}_a z = \frac{1}{b} \text{Ln}\, z, \tag{11.34}$$

which differs from Ln z only by the factor $1/b$. Since b is a value of Ln a, it follows from (11.34) that

$$\text{Log}_a z = \frac{\text{Ln}\, z}{\text{Ln}\, a}, \tag{11.35}$$

where in the denominator we fix one of the infinitely many values of Ln a (which is then kept the same for all z). Thus, to define $\text{Log}_a z$, we must specify not only the base a, but also a particular value of Ln a.

Example 1. Let $a = e$, and choose the value of Ln e equal to 1. Then

$$\text{Log}_e z = \text{Ln}\, z,$$

which is just the ordinary definition of the *natural logarithm*. However, we can also choose another value of Ln e, say $1 + 2\pi i$. In this case, (11.35) gives

$$\text{Log}_e z = \frac{\text{Ln}\, z}{1 + 2\pi i}.$$

It is easy to see that with this second definition of the natural logarithm, $\text{Log}_e z$ will have a real value (in fact, exactly one) only if $z = e^k$, where k is an integer.

Example 2. Let $a = 10$, and choose the value of Ln 10 equal to

$$2.30259\ldots = \frac{1}{0.43429\ldots} = \frac{1}{M}.$$

Then we have

$$\text{Log}_{10} z = M \,\text{Ln}\, z = (0.43429\ldots)\, \text{Ln}\, z. \tag{11.36}$$

This definition of the *common logarithm* of an arbitrary complex number $z \neq 0$ agrees with the ordinary definition of the common logarithm $\log_{10} x$ of a positive real number x. In fact, setting $z = x$ in (11.36) and taking principal values, we obtain

$$\log_{10} x = (0.43429\ldots) \ln x.$$

Example 3. If $a = 1$, we cannot define $\text{Log}_1 z$ by using the principal value of Ln 1, since $\ln 1 = 0$. Instead, suppose we choose the value of Ln 1 equal to $2\pi i$, obtaining

$$\text{Log}_1 z = \frac{\text{Ln}\, z}{2\pi i} = \frac{1}{2\pi} \text{Arg}\, z - \frac{i}{2\pi} \ln |z|.$$

With this definition, all the values of $\text{Log}_1 z$ are real if $|z| = 1$ and imaginary if $|z| \neq 1$. Thus, only numbers corresponding to points on the unit circle have real numbers as their logarithms to the base 1, and for such numbers $\text{Log}_1 z = \text{Arg } z$.

58. The Mapping $w = \text{Arc cos } z$

First we study the function $w = \text{Arc cos } z$, defined as the (multiple-valued) inverse of the function

$$z = \cos w.$$

Replacing $\cos w$ by

$$\frac{e^{iw} + e^{-iw}}{2},$$

and writing

$$e^{iw} = t \tag{11.37}$$

for brevity (cf. Sec. 41), we find that t satisfies the equation

$$\frac{1}{2}\left(t + \frac{1}{t}\right) = z$$

or

$$t^2 - 2zt + 1 = 0, \tag{11.38}$$

with solution[11]

$$t = z + \sqrt{z^2 - 1}. \tag{11.39}$$

It follows from (11.37) and (11.39) that

$$w = \text{Arc cos } z = -i \text{ Ln } t = -i \text{ Ln } (z + \sqrt{z^2 - 1}). \tag{11.40}$$

We now investigate the branch points of the multiple-valued function $w = \text{Arc cos } z$, which, according to (11.40), involves both a square root and a logarithm. Therefore we first examine the behavior of (11.40) at ± 1 (the branch points of $\sqrt{z^2 - 1}$), and then at $t = 0$ and $t = \infty$ (the branch points of $\text{Ln } t$).

1. *The function* Arc cos z *has algebraic branch points at* ± 1. In fact, as z makes one circuit around any closed Jordan curve whose interior contains either of the points ± 1 (but not the other), each value of $\sqrt{z^2 - 1}$ is replaced by its negative. Therefore each root

$$t = z + \sqrt{z^2 - 1}$$

of the quadratic equation (11.38) is replaced by the other root t^{-1} (recall that the product of the two roots is 1). Correspondingly, the expression $-i \text{ Ln } t$ is replaced by $-i \text{ Ln } t^{-1}$, which differs from the first expression if $t \neq t^{-1}$. But $t = t^{-1}$ implies $t = \pm 1$ and hence $z = \pm 1$, as can be seen from (11.38) or (11.39). This case cannot occur, since by hypothesis our Jordan curve does not pass through either of the points $z = \pm 1$. Therefore the value of Arc cos z actually does change as a result of the circuit around $z = 1$ or $z = -1$, and hence the points $z = \pm 1$ are both branch points, in fact, branch points of order 1.

2. *The function* Arc cos z *has a logarithmic branch point at* ∞. According to Sec. 51, t makes a circuit around a circle in the t-plane with center $t = 0$ if and only if z makes a circuit around an ellipse in the z-plane with foci at $z = \pm 1$. Thus,as the point z traces out an ellipse with foci ± 1, $\text{Arg } t$ changes by $\pm 2\pi$, and hence so does $-i \text{ Ln } t$. Since any neighborhood of the point at infinity contains an ellipse with foci ± 1, the point ∞ is a branch point of Arc cos z, in fact, a branch point of infinite order.[12]

3. *The function* Arc cos z *has no branch points in the extended z-plane except those already indicated, i.e.,* ± 1 *and* ∞. In fact, if $z_0 \neq \pm 1$, $z_0 \neq \infty$, then, corresponding to the value $z = z_0$, equation (11.38) has two roots t_0' and t_0'', which satisfy the condition $t_0' t_0'' = 1$, and are different from 0, ± 1, and ∞. We can find disjoint neighborhoods $\mathcal{N}(t_0')$ and $\mathcal{N}(t_0'')$ which are so small that they do not contain the points 0 and ∞, and are such that neither neighborhood contains a pair of points t_1, t_2 satisfying the condition $t_1 t_2 = 1$. To see this, let γ be the unit circle $|t| = 1$, and suppose that $t_0' \notin \gamma$, say $|t_0'| < 1$. Then $|t_0''| > 1$ and hence, if we choose $\mathcal{N}(t_0') \subset I(\gamma)$ and $\mathcal{N}(t_0'') \subset E(\gamma)$, neither neighborhood contains a pair of points t_1, t_2 such that $t_1 t_2 = 1$ (see Figure 11.6). On the other hand, if $t_0' \in \gamma$, then, since $t_0' \neq \pm 1$, t_0' belongs to either the upper half-plane $\Pi_U : \text{Im } t > 0$ or the lower half-plane $\Pi_L : \text{Im } t < 0$, say to Π_L. Then $t_0'' \in \gamma \cap \Pi_U$, and in this case it is sufficient to choose $\mathcal{N}(t_0') \subset \Pi_L$, $\mathcal{N}(t_0'') \subset \Pi_U$.

In any case, the function

$$z = \frac{1}{2}\left(t + \frac{1}{t}\right) \tag{11.41}$$

is one-to-one on each of the neighborhoods $\mathcal{N}(t_0')$ and $\mathcal{N}(t_0'')$, since it can only take the same value at a pair of points t_1, t_2 satisfying the

[11] We do not write $\pm$ in front of the radical, since the square root is already understood to be double-valued [cf. (9.52)]. Note that both numbers (11.39) are nonzero (in fact, their product equals 1).

[12] A circuit around any ellipse with foci at the points $z = \pm 1$ can also be regarded as a circuit around the point $z = 0$. However, it should be noted that none of these ellipses lies entirely in the neighborhood $|z| < \varepsilon$ if $\varepsilon \leqslant 1$, and in fact it turns out that $z = 0$ is not a branch point of Arc cos z.