Proof. We have

$$\delta = \ln(-R, R, b', 0) = \ln\left(\frac{b'+R}{b'-R} : \frac{0+R}{0-R}\right) = \ln\frac{R+b'}{R-b'},$$

where $b' > 0$ is the abscissa of the point B'. It follows that

$$\frac{R+b'}{R-b'} = e^{\delta}, \qquad b' = R\frac{e^{\delta}-1}{e^{\delta}+1}.$$

Suppose the tangent to Γ at the point β' intersects the real axis in the point C' [see Figure 10.9(b)]. Then C' is obviously the center of the circle containing the arc l', and $C'\beta' = \rho$ is the radius of this circle.[26] Moreover, $A'\beta'$ is tangent to the arc $\alpha'\beta'$ at β', and hence

$$(A'\beta')^2 = A'B'(A'B' + 2\rho). \tag{10.39}$$

Solving (10.39) for ρ, we obtain

$$\rho = \frac{R^2 - b'^2}{2b'} = \frac{R^2 - R^2[(e^{\delta}-1)/(e^{\delta}+1)]^2}{2R(e^{\delta}-1)/(e^{\delta}+1)} = \frac{2Re^{\delta}}{e^{2\delta}-1}$$
$$= \frac{R}{(e^{\delta}-e^{-\delta})/2} = \frac{R}{\sinh\delta}.$$

Finally, examining the triangle $A'\beta'C'$, we find that

$$\Pi(\delta) = \arctan\frac{C'\beta'}{A'\beta'} = \arctan\frac{\rho}{R} = \arctan\frac{1}{\sinh\delta},$$

as asserted.

Remark. It follows from (10.38) that the angle of parallelism satisfies the inequality

$$0 < \Pi(\delta) < \frac{\pi}{2}.$$

Moreover, as $\delta \to 0$,

$$\sinh\delta \to 0, \quad \frac{1}{\sinh\delta} \to +\infty, \quad \Pi(\delta) \to \frac{\pi}{2},$$

while as $\delta \to \infty$,

$$\sinh\delta \to +\infty, \quad \frac{1}{\sinh\delta} \to 0, \quad \Pi(\delta) \to 0.$$

Formula (10.38) plays a basic rôle in Lobachevskian *trigonometry*.

[26] Here, we also use $C'\beta'$, $A'\beta'$ and $A'B'$ to denote the lengths of the corresponding *Euclidean* line segments.

For the further development of the material presented in this section, we refer the reader to the extensive literature on non-Euclidean geometry.[27]

51. The Mapping $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

A mapping by a rational function of order higher than 1 was studied in Sec. 37, in connection with the function

$$w = (z-a)^n \qquad (n > 1).$$

However, this function is not meromorphic in the full sense of the word, since it is actually entire. We now study the rational function

$$w = \lambda(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}, \tag{10.40}$$

which comes up in the course of solving a variety of problems. In fact, because of the use which the Russian scientist Joukowski (1847–1921) made of this function in aerodynamics, it is often referred to as the *Joukowski function*. Obviously, $w = \lambda(z)$ is a rational function of order 2, which does not reduce to an entire function and which satisfies the condition

$$\lambda(z) = \lambda\left(\frac{1}{z}\right). \tag{10.41}$$

It follows from (10.41) that under the mapping $w = \lambda(z)$ every point of the w-plane except $w = \pm 1$ has two (and only two) distinct inverse images z_1 and z_2 satisfying the relation

$$z_1 z_2 = 1. \tag{10.42}$$

Now let γ be the unit circle $|z| = 1$, with interior $I(\gamma)$ and exterior $E(\gamma)$. Then, according to (10.42), $z_1 \in I(\gamma)$ if and only if $z_2 \in E(\gamma)$. Moreover, the two sets $\lambda[I(\gamma)]$ and $\lambda[E(\gamma)]$ are identical. Clearly, the function $w = \lambda(z)$ is continuous (in the wide sense) on the closed domains $\overline{I(\gamma)}$ and $\overline{E(\gamma)}$, and one-to-one on $I(\gamma)$ and $E(\gamma)$. Therefore, according to Theorem 6.1, $w = \lambda(z)$ maps $I(\gamma)$ and $E(\gamma)$ onto some domain $\mathscr{G}$ in the w-plane. Moreover, according to Theorem 6.3, to determine the boundary of this domain, we have to find the image $\Gamma = \lambda(\gamma)$ of the unit circle. But if

$$z = e^{it} \qquad (0 \leqslant t \leqslant 2\pi),$$

then

$$w = u + iv = \tfrac{1}{2}(e^{it} + e^{-it}) = \cos t \qquad (0 \leqslant t \leqslant 2\pi),$$

[27] See e.g., K. Borsuk and W. Smielew, *Foundations of Geometry*, revised edition (translated by E. Marquit), North-Holland Publishing Co., Amsterdam (1960); N. V. Efimov, *Höhere Geometrie*, VEB Deutscher Verlag der Wissenschaften, Berlin (1960); H. E. Wolfe, *Introduction to Non-Euclidean Geometry*, Holt, Rinehart and Winston, Inc., New York. (1945).

i.e., Γ is the segment $-1 \leqslant u \leqslant 1$, $v = 0$ of the real axis, traversed twice. It follows that the domain $\mathscr{G}$ consists of all points of the w-plane except those belonging to the segment Γ, i.e., $\mathscr{G} = \Gamma^c$, where Γ^c is the complement of Γ.

To study the mapping (10.40) in more detail, we find the images of the circles $|z| = r$ and the rays $\operatorname{Arg} z = \alpha + 2k\pi$ (see Figure 10.10), confining ourselves to the domain $G = I(\gamma)$. Setting

$$z = re^{it} \qquad (0 \leqslant t \leqslant 2\pi),$$

where $0 < r < 1$, we find that

$$w = u + iv = \frac{1}{2}\left(re^{it} + \frac{1}{r}e^{-it}\right) = \frac{1}{2}\left(\frac{1}{r} + r\right)\cos t - i\frac{1}{2}\left(\frac{1}{r} - r\right)\sin t,$$

or

$$u = \frac{1}{2}\left(\frac{1}{r} + r\right)\cos t, \qquad v = -\frac{1}{2}\left(\frac{1}{r} - r\right)\sin t, \tag{10.43}$$

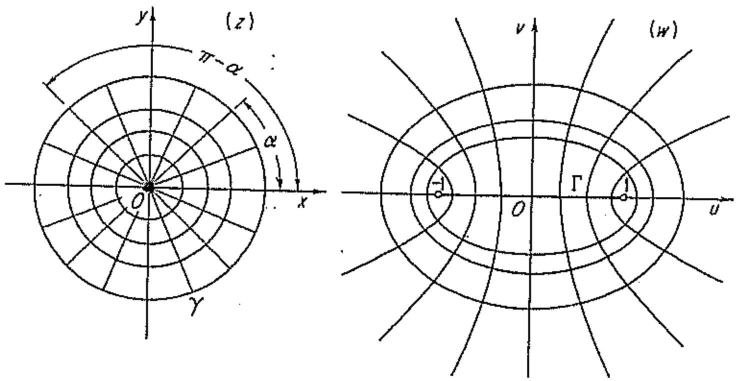

FIGURE 10.10

where $0 \leqslant t \leqslant 2\pi$. Eliminating t from (10.43), we obtain

$$\frac{u^2}{\left[\frac{1}{2}\left(\frac{1}{r} + r\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(\frac{1}{r} - r\right)\right]^2} = 1, \tag{10.44}$$

which is the equation of an ellipse with semiaxes $a = \frac{1}{2}(r^{-1} + r)$, $b = \frac{1}{2}(r^{-1} - r)$ and foci at ± 1. It follows from (10.43) that as t increases continuously from 0 to 2π, i.e., as the point z describes the whole circle $|z| = r$ once in the counterclockwise direction, the image point w describes the whole ellipse once in the clockwise direction. By varying the radius of the circle $|z| = r$ from 0 to 1, we cause a to decrease from ∞ to 1 and b to decrease from ∞ to 0; as a result, the ellipses (10.44) range over the whole set of ellipses with foci ± 1. It follows, without recourse to the general considerations of Chap. 6, that $w = \lambda(z)$ is a one-to-one mapping of the unit disk $G = I(\gamma)$ onto the domain $\mathscr{G} = \Gamma^c$.

Next we consider the images of the ray

$$z = te^{i\alpha} \qquad (0 \leqslant t < 1) \tag{10.45}$$

with inclination α.[28] Substituting (10.45) into (10.40), we obtain

$$w = u + iv = \frac{1}{2}\left(\frac{1}{t} + t\right)\cos\alpha - i\frac{1}{2}\left(\frac{1}{t} - t\right)\sin\alpha$$

or

$$u = \frac{1}{2}\left(\frac{1}{t} + t\right)\cos\alpha, \qquad v = -\frac{1}{2}\left(\frac{1}{t} - t\right)\sin\alpha \qquad (0 \leqslant t < 1). \tag{10.46}$$

It follows that the images of two radii symmetric with respect to the real axis (i.e., such that if one radius has inclination α, the other has inclination $-\alpha$) are themselves symmetric with respect to the real axis, while the images of two radii symmetric with respect to the imaginary axis (i.e., such that if one radius has inclination α, the other has inclination $\pi - \alpha$) are themselves symmetric with respect to the imaginary axis. Thus it is sufficient to consider the images of radii lying in the first quadrant $0 \leqslant \alpha \leqslant \pi/2$.

For $\alpha = 0$ we have

$$u = \frac{1}{2}\left(\frac{1}{t} + t\right), \quad v = 0 \qquad (0 \leqslant t < 1),$$

which represents the infinite interval $1 < u \leqslant +\infty$. The infinite interval $-\infty \leqslant u < -1$ is the image of the radius with inclination π. For $\alpha = \pi/2$ we have

$$u = 0, \quad v = -\frac{1}{2}\left(\frac{1}{t} - t\right) \qquad (0 \leqslant t < 1),$$

which represents the negative imaginary axis $-\infty \leqslant v < 0$. The positive imaginary axis $0 < v \leqslant +\infty$ is the image of the radius with inclination $\alpha = -\pi/2$. Thus the image of the "horizontal" diameter of the unit disk $G: |z| < 1$ is the infinite interval of the real axis going from -1 to $+1$ through the point at infinity and excluding the points ± 1, while the image of the "vertical" diameter of G is the whole imaginary axis including the point at infinity but excluding the origin.

Suppose now that $0 < \alpha < \pi/2$. Then, eliminating the parameter t from (10.46), we obtain

$$\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 1,$$

which is the equation of a hyperbola, which we denote by H, with semiaxes $a = \cos\alpha$, $b = \sin\alpha$ and foci at ± 1. Let H_1, H_2, H_3 and H_4 denote the intersections of H with the first, second, third and fourth quadrants,

[28] By the *inclination* of a ray (or directed line segment), we mean the angle measured from the positive real axis to the ray. The slope of the ray equals the tangent of its inclination.

respectively, excluding the two points $(\pm a, 0)$ of H belonging to the real axis. Moreover, let R_α, $R_{\pi-\alpha}$, $R_{\pi+\alpha}$ and $R_{-\alpha}$ denote the sets of points belonging to the radii (10.45), with inclinations α, $\pi - \alpha$, $\pi + \alpha$, and $-\alpha$, respectively. Then it is easy to see that

$$H_1 = \lambda(R_{-\alpha}), \quad H_2 = \lambda(R_{\pi+\alpha}), \quad H_3 = \lambda(R_{\pi-\alpha}), \quad H_4 = \lambda(R_\alpha).$$

In particular, the image of each of the diameters $R_\alpha \cup R_{\pi+\alpha}$ and $R_{\pi-\alpha} \cup R_{-\alpha}$ of G is a set consisting of two "quarter-branches" of the hyperbola H, joined at infinity and minus the points $(\pm a, 0)$.

To summarize, *the function*

$$w = \lambda(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

is a one-to-one continuous mapping of both the interior and the exterior of the unit circle γ *onto the complement of the segment* $-1 \leqslant u \leqslant +1$ *of the real axis. Under this mapping, the one-parameter family of circles* $|z| = r$ $(0 < r < 1)$ *is transformed into a one-parameter family of confocal ellipses, with semiaxes* $\frac{1}{2}(r^{-1} \pm r)$ *and foci at* ± 1, *and the one-parameter family of pairs of diameters of* γ *symmetric with respect to the real axis, formed of the radii*

$$z = \pm te^{\pm i\alpha} \qquad (0 \leqslant t < 1),$$

where $0 < \alpha < \pi/2$,[29] *is transformed into a one-parameter family of confocal hyperbolas (minus their vertices), with semiaxes* $\cos\alpha$, $\sin\alpha$ *and foci at* ± 1.

Remark. Since the derivative

$$\lambda'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

is nonzero for $z \neq 1$, the mapping is conformal at all points of the domains $I(\gamma)$ and $E(\gamma)$. It follows that the hyperbolas intersect the ellipses at the same angles as the radii intersect the circles, i.e., at right angles. A similar situation has already been encountered in Sec. 41.

We now study the images under $w = \lambda(z)$ of circles passing through the points $z = \pm 1$. It follows from

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z} \tag{10.47}$$

that

$$w - 1 = \frac{z^2 - 2z + 1}{2z} = \frac{(z-1)^2}{2z},$$

$$w + 1 = \frac{z^2 + 2z + 1}{2z} = \frac{(z+1)^2}{2z},$$

and hence

$$\frac{w - 1}{w + 1} = \left(\frac{z - 1}{z + 1}\right)^2. \tag{10.48}$$

[29] The cases $\alpha = 0$ and $\alpha = \pi/2$ warrant special discussion (see above).

Moreover, it is easy to see that (10.48) implies (10.47). Therefore we see that the mapping $w = \lambda(z)$ is the result of making the following three mappings in succession:

$$\tilde{z} = \frac{z - 1}{z + 1}, \quad \tilde{w} = \tilde{z}^2, \quad w = \frac{1 + \tilde{w}}{1 - \tilde{w}}. \tag{10.49}$$

The first of these mappings carries any circle γ (not necessarily the unit circle) passing through the points ± 1 into a straight line passing through the origin, the second mapping carries the straight line into a ray emanating from the origin, and the third mapping carries the ray into a circular arc δ joining the points ± 1. From (10.49) we also see that if the angle between γ and the positive real axis at the point $z = 1$ equals θ, then the angle between its image δ and the positive real axis at the point $w = 1$ equals 2θ (see Figure 10.11). Moreover, it is clear that each of the two arcs of γ with end points ± 1 is separately mapped into the same arc δ.

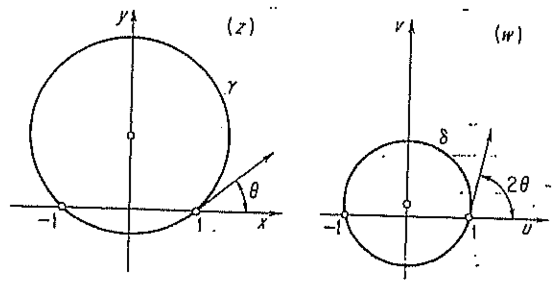

FIGURE 10.11

Remark 1. The first of the mappings (10.49) transforms the exterior of the circle γ into a half-plane, the second mapping transforms the half-plane into a domain whose boundary is a ray emanating from the origin, and the third transforms this domain into a domain whose boundary is the arc δ. Since all three mappings are one-to-one and conformal on the appropriate domains, the function $w = \lambda(z)$ is a one-to-one conformal mapping of the exterior of the circle γ (and also of its interior) onto a domain whose boundary is the circular arc δ joining the points ± 1. This generalizes the case where γ is the unit circle, and δ is the line segment joining -1 to $+1$.

Remark 2. Let Π_U be the upper half-plane $\operatorname{Im} z > 0$, let Π_L be the lower half-plane $\operatorname{Im} z < 0$, and let K be the unit disk $|z| < 1$. Then the function $w = \lambda(z)$ maps $K_U = K \cap \Pi_U$ onto Π_L and $K_L = K \cap \Pi_L$ onto Π_U. Moreover

$$\lambda(\Pi_U - \bar{K}_U) = \lambda(K_L) = \Pi_U,$$

since $\lambda(z) = \lambda(1/z)$. The image of the semicircle γ separating $\Pi_U - \bar{K}_U$ from K_U is the segment $\Gamma: -1 \leqslant u \leqslant +1$, traversed just once. Therefore the image of the upper half-plane under $w = \lambda(z)$ is the whole plane except for the infinite segment of the real axis which joins -1 to $+1$ and passes through the point at infinity, with K_U going into Π_L and $\Pi_U - \bar{K}_U$ going into Π_U, as shown in Figure 10.12.

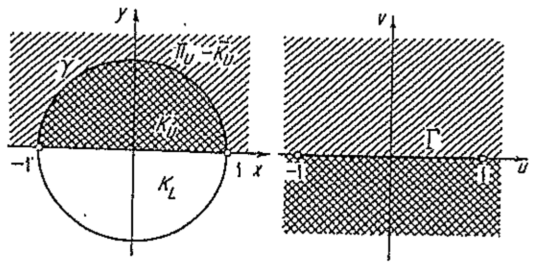

FIGURE 10.12

52. Transcendental Meromorphic Functions. Trigonometric Functions.

In this section, we study the simplest *transcendental meromorphic functions*, by which we mean meromorphic functions which are not rational functions. For example, the meromorphic functions

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z} \qquad (10.50)$$

are all transcendental, since, unlike rational functions, they have infinitely many poles (points where the functions become infinite). The functions (10.50), as well as the entire functions $\cos z$ and $\sin z$, belong to the class of *trigonometric functions*, i.e., meromorphic functions of the form[29]

$$f(z) = \frac{P(e^{iz})}{Q(e^{iz})} = \frac{a_0 + a_1e^{iz} + \cdots + a_me^{imz}}{b_0 + b_1e^{iz} + \cdots + b_ne^{inz}}. \qquad (10.51)$$

Thus a trigonometric function is a rational function of the complex exponential e^{iz}. Obviously, we obtain the same class by considering functions which can be represented in the form

$$F(z) = \frac{\sum_{j=-m}^{m} a_je^{ijz}}{\sum_{k=-n}^{n} b_ke^{ikz}}.$$

[29] In Sec. 40 we studied the two particularly simple *entire* trigonometric functions $\cos z$ and $\sin z$.

This last expression can be written as

$$F(z) = \frac{a_0 + \sum_{j=1}^{m} [(a_j + a_{-j}) \cos jz + i(a_j - a_{-j}) \sin jz]}{b_0 + \sum_{k=1}^{n} [(b_k + b_{-k}) \cos kz + i(b_k - b_{-k}) \sin kz]}$$
$$= \frac{a_0 + \sum_{j=1}^{m} (A_j' \cos jz + A_j'' \sin jz)}{b_0 + \sum_{k=1}^{n} (B_k' \cos kz + B_k'' \sin kz)}.$$

In the special case where the denominator is a constant, we can set $b = 1$, obtaining a *trigonometric polynomial*

$$F(z) = a_0 + \sum_{j=1}^{m} (A_j' \cos jz + A_j'' \sin jz).$$

If at least one of the numbers A_m' and A_m'' is nonzero, $F(z)$ is said to be a trigonometric polynomial of *degree m*.

Obviously, every trigonometric function is periodic with period 2π. Therefore it is sufficient to study its behavior in any strip $G: x_0 \leqslant x < x_0 + 2\pi$ parallel to the imaginary axis (note that G is neither open nor closed), since the function behaves in exactly the same way in all the strips

$$G_k: x_0 + 2k\pi \leqslant x < x_0 + (2k + 1)\pi,$$

where $k = 0, \pm 1, \pm 2, \ldots$ and $G_0 = G$. As z ranges over the strip G (including the line $x = x_0$), the variable $z_1 = iz$ ranges over a strip $x_0 \leqslant y < x_0 + 2\pi$ of the same width 2π parallel to the real axis, and hence $t = e^{iz}$ describes an angle of 2π radians with its vertex at the origin. The sides of this angle coalesce into a single ray $\operatorname{Arg} t = x_0 + 2k\pi$, which is swept out as z moves along the line $\operatorname{Re} z = x_0$. Therefore, as z ranges over the whole strip G, $t = e^{iz}$ ranges over the whole plane, taking every value except $t = 0$ and $t = \infty$ (recall that e^z does not approach a limit as $z \to \infty$).

Returning to the function (10.51), we write

$$R(t) = \frac{P(t)}{Q(t)},$$

so that

$$f(z) = R(e^{iz}).$$

Suppose the polynomials $P(t)$ and $Q(t)$ are of degrees m and n, respectively, so that the rational function $R(t)$ is of order $N = \max(m, n)$. Then, as we know from Sec. 43, for every complex number A, the equation

$$R(t) = A \qquad (10.52)$$

has N roots in the extended t-plane, and there are at most $m + n$ values of A for which (10.52) can have multiple roots. However, in our case, the values $t = 0$ and $t = \infty$ are excluded, since they are not possible values of the function $t = e^{iz}$. Therefore we can only assert that (10.52) has N roots (some of which may be multiple) for every value of A with the possible exception of the numbers $R(0)$ and $R(\infty)$. Since the correspondence between z and t is one-to-one on the strip G, and since the behavior of the function $z = e^{it}$ is identical on all the strips G_k, where $k = 0, \pm 1, \pm 2, \ldots$ and $G_0 = G$, we have proved the following result:

THEOREM 10.12. *If*

$$f(z) = R(e^{iz}) \not\equiv \text{const}$$

is a trigonometric function, the equation

$$f(z) = A$$

has infinitely many roots for every complex number A with the possible exception of $R(0)$ and $R(\infty)$, for which the equation may have no roots at all.

Example 1. Let

$$f(z) = e^{inz},$$

so that

$$R(t) = t^n,$$

where $t = e^{iz}$. Then $R(0) = 0$, $R(\infty) = \infty$, and moreover the function $R(t)$ does not vanish or become infinite unless $t = 0$ or $t = \infty$. Therefore $f(z)$ does not vanish or become infinite for any z.

Example 2. Let

$$f(z) = \sec z = \frac{2e^{iz}}{1 + e^{2iz}},$$

so that

$$R(t) = \frac{2t}{1 + t^2}.$$

Then $R(0) = R(\infty) = 0$, and moreover the function $R(t)$ does not vanish unless $t = 0$ or $t = \infty$. Therefore $f(z)$ does not vanish for any value of z.

Example 3. Let

$$f(z) = \tan z = \frac{1}{i}\frac{e^{2iz} - 1}{e^{2iz} + 1},$$

so that

$$R(t) = \frac{1}{i}\frac{t^2 - 1}{t^2 + 1}.$$

Since $R(0) = i$ and $R(\infty) = -i$, and since $R(t)$ does not equal $\pm i$ unless $t = 0$ or $t = \infty$, the function $f(z)$ does not equal $\pm i$ for any z.

Example 4. Let

$$f(z) = \frac{\cos z}{\sin 2z} = i\frac{e^{3iz} + e^{iz}}{e^{4iz} - 1},$$

so that

$$R(t) = i\frac{t^3 + t}{t^4 - 1}$$

and $R(0) = R(\infty) = 0$. However, $R(t)$ vanishes at points other than 0 and ∞, i.e., for $t = \pm i$. Therefore, in this case, $f(z)$ takes *all* complex values at infinitely many points.

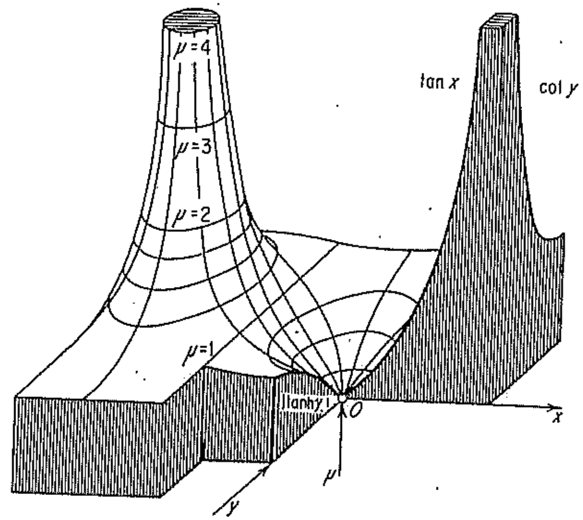

FIGURE 10.13

We now study the function

$$f(z) = \tan z = \frac{1}{i}\frac{t^2 - 1}{t^2 + 1} \qquad (t = e^{iz})$$

in more detail. In Figure 10.13 we plot the *modular surface* of tan z, i.e., the surface $\mu = |\tan z|$.[30] The mapping $w = \tan z$ can be regarded as the result of making the following four mappings in succession:

$$\zeta = iz, \quad t = e^{\zeta}, \quad \tau = t^2, \quad w = \frac{1}{i}\frac{\tau - 1}{\tau + 1}. \tag{10.53}$$

[30] The unlabelled curves correspond to the equation arg tan z = const.

If G is the strip

$$x_0 < x < x_0 + h \qquad (0 < h \leqslant \pi) \tag{10.54}$$

parallel to the real axis, the mapping $\zeta = iz = \xi + i\eta$ carries G into the strip

$$x_0 < \eta < x_0 + h \tag{10.55}$$

parallel to the imaginary axis. Then the mapping $t = e^{\zeta}$ carries the strip (10.55) into the interior of the angle of h radians with vertex at the origin and sides

$$\operatorname{Arg} t = x_0 + 2k\pi, \quad \operatorname{Arg} t = x_0 + h + 2l\pi \qquad (k, l = 0, \pm 1, \pm 2, \ldots).$$

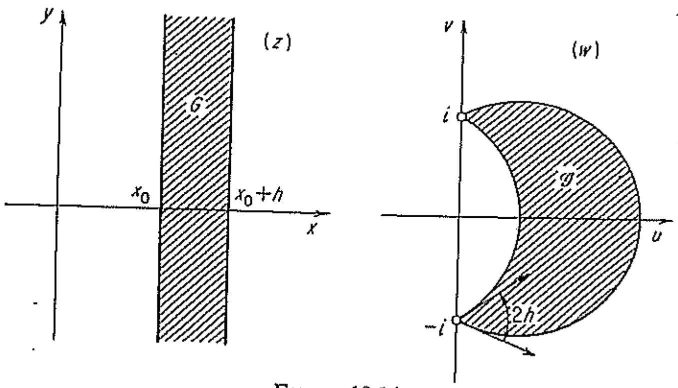

FIGURE 10.14

Under the mapping $\tau = t^2$, this angle is carried into the domain A which is the interior of the angle of $2h$ radians with sides

$$\operatorname{Arg} \tau = 2x_0 + 2k\pi, \qquad \operatorname{Arg} \tau = 2x_0 + 2h + 2l\pi, \tag{10.56}$$

and finally, the Möbius transformation

$$w = \frac{1}{i}\frac{\tau - 1}{\tau + 1} \tag{10.57}$$

carries the rays (10.56) into circular arcs joining the points $w = i$ $(\tau = 0)$ and $w = -i$ $(\tau = \infty)$, while carrying A itself into a circular lune with angles of $2h$ radians and vertices $\pm i$ (see Figure 10.14). Since

$$\left.\frac{dw}{d\tau}\right|_{\tau=0} = -2i, \qquad \operatorname{Arg}(-2i) = -\frac{\pi}{2} + 2k\pi,$$

the mapping (10.57) rotates the tangent to any curve emanating from the point $\tau = 0$ through the angle $-\frac{1}{2}\pi$. Therefore, at the point $w = i$, the tangents to the circular arcs bounding the lune must have inclinations $2x_0 - \frac{1}{2}\pi + 2k\pi$ and $2x_0 + 2h - \frac{1}{2}\pi + 2l\pi$. These conditions completely determine the domain $\mathscr{G} = f(G)$.

Remark. Each of the mappings (10.53) is one-to-one and conformal on the appropriate domains. It follows that $w = \tan z$ is a one-to-one conformal mapping of the strip (10.54) onto a circular lune with angles $2h$ and vertices $\pm i$. In particular, if $h = \pi/2$, the angles of the lune become π, and the lune itself becomes a circle.

PROBLEMS

10.1 Given a rational function

$$f(z) = \frac{P(z)}{Q(z)}, \tag{10.58}$$

prove that the mapping $w = f(z)$ is conformal at any simple zero of $Q(z)$, and also at $z = \infty$ if the equation $f(z) = f(\infty)$ has no multiple roots.

10.2. Prove that the mapping (10.58) actually fails to be conformal at each of the points $\gamma_1, \ldots, \gamma_r$ which are the roots of the equation

$$P'(z)Q(z) - P(z)Q'(z) = 0$$

[see (10.8)], and also at the point $\gamma_0 = \infty$ if the equation $f(z) = f(\infty)$ has multiple roots. In particular, prove that an angle with vertex at γ_j is enlarged a number of times equal to the multiplicity of the root γ_j of the equation $f(z) = f(\gamma_j)$, $j = 0, 1, \ldots, r$.

10.3. Prove that the Möbius transformations of the special form

$$L_1 = z, \qquad L_2 = \frac{1}{z}, \qquad L_3 = 1 - z,$$
$$L_4 = \frac{1}{1 - z}, \quad L_5 = \frac{z - 1}{z}, \quad L_6 = \frac{z}{z - 1} \tag{10.59}$$

form a group.

Comment. This fact is summarized by saying that the transformations (10.59) are a *subgroup* of $\mathscr{M}$, the group of Möbius transformations (see footnote 13, p. 184).

10.4. Find the images of the following domains under the indicated Möbius transformations:

a) The quadrant $x > 0$, $y > 0$ if $w = \dfrac{z - i}{z + i}$;

b) The half-disk $|z| < 1$, $\operatorname{Im} z > 0$ if $w = \dfrac{2z - i}{2 + iz}$;

c) The sector $0 < \arg z < \dfrac{\pi}{4}$ if $w = \dfrac{z}{z - 1}$;

d) The strip $0 < x < 1$ if $w = \dfrac{z - 1}{z}$ or if $w = \dfrac{z - 1}{z - 2}$.