1. $\mathscr{M}$ *is closed under multiplication*, i.e., if $L_1 \in \mathscr{M}$, $L_2 \in \mathscr{M}$, then $L_1L_2 \in \mathscr{M}$, $L_2L_1 \in \mathscr{M}$.
2. Multiplication is associative.
3. There is an element $U \in \mathscr{M}$ such that $LU = UL = L$ for any $L \in \mathscr{M}$.
4. For each $L \in \mathscr{M}$, there is an element $L^{-1} \in \mathscr{M}$ such that $LL^{-1} = L^{-1}L = U$.

In algebraic language, these four properties are summarized by saying that $\mathscr{M}$ *is a group of transformations.*[6]

45. The Circle-Preserving Property of Möbius Transformations

We now prove that any Möbius transformation carries a straight line or a circle into another straight line or circle. We call this the *circle-preserving property*, since a straight line can be regarded as a limiting case of a circle (corresponding to infinite radius). The entire linear transformation $L(z) = \alpha z + \beta$ $(\alpha \neq 0)$ is obviously circle-preserving, since the mapping $w = L(z)$ is just a shift (if $\alpha = 1$), or a shift combined with a rotation and a uniform magnification (if $\alpha \neq 1$), as discussed in Sec. 33.

LEMMA. *The transformation*

$$w = \Lambda(z) = \frac{1}{z} \tag{10.15}$$

is circle-preserving.

Proof. The equation of any straight line or circle in the z-plane can be written in the form

$$A(x^2 + y^2) + 2Bx + 2Cy + D = 0, \tag{10.16}$$

where we have a straight line if $A = 0$ and at least one of the numbers B, C is nonzero, and a circle if $A \neq 0$ and $B^2 + C^2 - AD > 0$. Since

$$x^2 + y^2 = z\bar{z}, \quad 2x = z + \bar{z}, \quad 2y = -i(z - \bar{z}),$$

where $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$, we can rewrite (10.16) as

$$Az\bar{z} + \bar{E}z + E\bar{z} + D = 0, \tag{10.17}$$

where $E = B + iC$. It is easy to see that equation (10.17), where A and D are real and E is complex, is the equation of a straight line if and only if $A = 0$, $E \neq 0$, and the equation of a circle if and only if $A \neq 0$, $E\bar{E} - AD > 0$.

[6] See e.g., G. Birkhoff and S. MacLane, *op. cit.*, Chap. 6, Sec. 2, or V. I. Smirnov, *Linear Algebra and Group Theory* (translated by R. A. Silverman), McGraw-Hill Book Co., New York (1961), Sec. 62.

We now find the image of the curve with equation (10.17) under the transformation (10.15). Replacing z by $1/w$ in (10.17), we obtain

$$A\frac{1}{w\bar{w}} + \bar{E}\frac{1}{w} + E\frac{1}{\bar{w}} + D = 0$$

or

$$Dw\bar{w} + Ew + \bar{E}\bar{w} + A = 0. \tag{10.18}$$

Equation (10.18) has the same form as equation (10.17), with D, $\bar{E}$ and A substituted for A, E and D, respectively. It follows that (10.18) is the equation of a straight line if $D = 0$, since then either $A = 0$ and $E \neq 0$ if (10.17) is the equation of a straight line, or else $A \neq 0$ and $E\bar{E} - AD = E\bar{E} > 0$ (so that $E \neq 0$ again) if (10.17) is the equation of a circle. Moreover, (10.18) is the equation of a circle if $D \neq 0$, since then either $A \neq 0$ and $E\bar{E} - AD > 0$ if (10.17) is the equation of a circle, or else $A = 0$ and $E \neq 0$ (so that $E\bar{E} - AD = E\bar{E} > 0$ again) if (10.17) is the equation of a straight line.

THEOREM 10.4. *Every Möbius transformation*

$$w = L(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{10.19}$$

is circle-preserving.

Proof. If $c = 0$, (10.19) reduces to an entire linear transformation and hence is circle-preserving. If $c \neq 0$, (10.19) can be written in the form

$$w = \frac{a}{c} + \frac{bc - ad}{c(cz + d)}.$$

Setting

$$z_1 = L_1(z) = cz + d, \quad z_2 = \Lambda(z_1) = \frac{1}{z_1},$$
$$w = L_2(z_2) = \frac{a}{c} + \frac{bc - ad}{c} z_2,$$

we can write $L(z)$ as a product

$$L = L_2 \Lambda L_1$$

of three transformations which are all circle-preserving (use the lemma). It follows that L itself is circle-preserving.

COROLLARY. *Let* $\delta = -d/c$ *be the pole of the function* (10.19). *Then* (10.19) *transforms every straight line or circle which passes through* δ *into a straight line, and every other straight line or circle into a circle.*

Proof. If the circle or straight line passes through δ, its image under (10.19) contains the point at infinity, and hence must be a straight line, since it cannot be a circle (no circle contains ∞). Similarly, if the circle or straight line does not pass through δ, its image does not contain the point at infinity, and hence must be a circle, since it cannot be a straight line (every straight line contains ∞).

Remark. Let $w = L(z)$ be any Möbius transformation, let γ be a straight line or circle in the z-plane, and let $\Gamma = L(\gamma)$ be the image of γ in the w-plane (Γ is itself a straight line or a circle). The two domains G_1 and G_2 with boundary γ are either two half-planes or the interior and exterior of a circle. Let $L(G_1)$ and $L(G_2)$ be the images of these two domains under the mapping $w = L(z)$. We now show that $L(G_1)$ and $L(G_2)$ are the two domains whose common boundary is the curve Γ.[7]

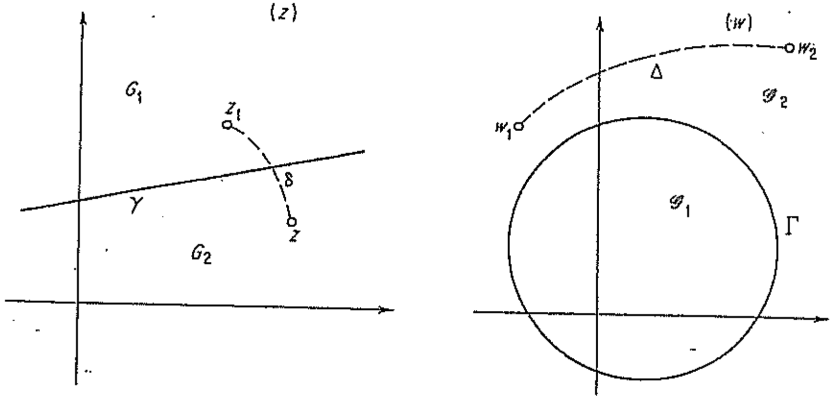

FIGURE 10.1

First suppose $z_1 \in G_1$, $z_2 \in G_2$, and let $w_1 = L(z_2)$, $w_2 = L(z_2)$. Then $w_1 \notin \Gamma$, $w_2 \notin \Gamma$, since $z_1 \notin \gamma$, $z_2 \notin \gamma$, and hence w_1 and w_2 must belong to the union of the two (disjoint) domains into which Γ divides the extended w-plane. If w_1 and w_2 both belong to one of these two domains, we can join w_1 to w_2 by a line segment or circular arc Δ which does not intersect Γ (see Figure 10.1). The inverse image of Δ in the z-plane must be a line segment or circular arc δ, which joins z_1 to z_2 and does not intersect γ. But the existence of δ contradicts the assumption that z_1 and z_2 belong to different domains G_1 and G_2. Therefore, if z_1 and z_2 belong to different domains with boundary γ, their images w_1 and w_2 must belong to different domains with boundary Γ.

We now denote the domains containing w_1 and w_2 by $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively. If z is an arbitrary point in G_1, then, since z and z_2 belong to different domains G_1 and G_2, their images w and w_2 belong to different

[7] Of course, this result follows at once from Theorem 6.3, knowledge of which is not presupposed here.

domains $\mathscr{G}_1$ and $\mathscr{G}_2$. But $w_2 \in \mathscr{G}_2$, and hence $w \in \mathscr{G}_1$, i.e., $L(z) \in \mathscr{G}_1$, if $z \in G_1$. Similarly, $L(z) \in \mathscr{G}_2$ if $z \in G_2$, and hence

$$\mathscr{G}_1 \supset L(G_1), \qquad \mathscr{G}_2 \supset L(G_2). \tag{10.20}$$

Conversely, let w be an arbitrary point in $\mathscr{G}_1$. Then w must be the image of a point z in G_1 or G_2. But $z \in G_2$ implies $w \in \mathscr{G}_2$, contrary to hypothesis, and hence $z \in G_1$, i.e., $\mathscr{G}_1 \subset L(G_1)$. Similarly, we find that $\mathscr{G}_2 \subset L(G_2)$. It follows by comparison with (10.20) that

$$\mathscr{G}_1 = L(G_1), \qquad \mathscr{G}_2 = L(G_2),$$

i.e., the two domains with boundary Γ are just the images of the two domains G_1 and G_2, as asserted. Moreover, to determine which of the two domains with boundary Γ is actually the image of a given domain G_1 with boundary γ, it is sufficient to locate the image w_1 of any point $z_1 \in G_1$, for then the domain $\mathscr{G}_1$ containing w_1 is the image of G_1.

46. Fixed Points of a Möbius Transformation. Invariance of the Cross Ratio

By a *fixed point* of a transformation or mapping $w = f(z)$, we mean a point which is carried into itself by the transformation. Obviously, every such point is a solution of the equation

$$z = f(z).$$

Moreover, every point of the z-plane is trivially a fixed point of the unit transformation $U(z) = z$.

THEOREM 10.5. *Every Möbius transformation different from the unit transformation has two fixed points, which in certain cases coalesce into a single fixed point.*

Proof. First let $c = 0$ ($d \neq 0$), so that $L(z)$ reduces to the entire linear transformation

$$L(z) = \alpha z + \beta \qquad \left(\alpha = \frac{a}{d},\ \beta = \frac{b}{d}\right).$$

Then, since $L(\infty) = \infty$, one fixed point is the point at infinity. If $\alpha \neq 1$, there exists another fixed point determined from the equation

$$z = \alpha z + \beta,$$

i.e., the point $\beta/(1 - \alpha)$, but if $\alpha = 1$, $\beta \neq 0$, there is no finite fixed point. Moreover, if $\alpha \neq 1$, $\beta \neq 0$, the finite fixed point $\beta/(1 - \alpha)$ approaches ∞ as $\alpha \to 1$. Therefore, in the case of the transformation

$$L(z) = z + \beta \qquad (\beta \neq 0),$$

the point at infinity can be regarded as two fixed points which have coalesced.

Now let $c \neq 0$, so that

$$L(\infty) = \frac{a}{c} \neq \infty,$$

i.e., the point at infinity is not a fixed point.[8] Similarly, the pole $\delta = -d/c$ of the transformation is not a fixed point, since

$$L(\delta) = \infty \neq \delta.$$

Assuming that $z \neq \infty$ and $z \neq \delta$, we solve the equation

$$z = \frac{az + b}{cz + d}$$

or

$$cz^2 - (a - d)z - b = 0,$$

obtaining

$$z = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

If $(a - d)^2 + 4bc \neq 0$, we obtain two different finite fixed points; if $(a - d)^2 + 4bc = 0$, these two points coalesce to form a single finite fixed point $(a - d)/2c$.

COROLLARY. *The only Möbius transformation with more than two fixed points is the unit transformation $U(z) = z$, for which all points are fixed points.*

THEOREM 10.6. *A sufficient condition for two Möbius transformations $L(z)$ and $\Lambda(z)$ to be identical is that the equation*

$$L(z) = \Lambda(z)$$

hold for three distinct points z_1, z_2 and z_3. In particular, there cannot exist two distinct Möbius transformations taking three given values w_1, w_2, w_3 at three given distinct points z_1, z_2, z_3.

Proof. It follows from

$$L(z_k) = \Lambda(z_k) = w_k \qquad (k = 1, 2, 3)$$

that

$$\Lambda^{-1}(w_k) = z_k \qquad (k = 1, 2, 3),$$

and hence

$$\Lambda^{-1}L(z_k) = z_k \qquad (k = 1, 2, 3).$$

[8] Therefore, in the class of Möbius transformations, an entire linear transformation is characterized by the fact that at least one of its fixed points is the point at infinity.

Therefore the transformation $\Lambda^{-1}L$ has three distinct fixed points, which, according to the corollary, implies

$$\Lambda^{-1}L = U,$$

where U is the unit transformation. Multiplying both sides of (10.21) by Λ from the left, we obtain

$$\Lambda(\Lambda^{-1}L) = \Lambda U,$$

which implies

$$L = \Lambda,$$

as asserted.

We now set about determining the (unique) Möbius transformation $w = L(z)$ carrying the points z_1, z_2, z_3 into the points w_1, w_2, w_3. First we consider the problem of finding the special Möbius transformation $w = \Lambda(z)$ carrying three finite points z_1, z_2, z_3 into the points $0, \infty, 1$. Since the function

$$w = \Lambda(z) = \frac{az + b}{cz + d}$$

vanishes for $z = z_1$ and becomes infinite for $z = z_2$ if and only if z_1 is a zero of the numerator and z_2 is a zero of the denominator, it follows that

$$w = \Lambda(z) = \frac{a}{c}\frac{z - z_1}{z - z_2}.$$

But w must equal 1 for $z = z_3$, i.e.,

$$1 = \frac{a}{c}\frac{z_3 - z_1}{z_3 - z_2},$$

which implies[9]

$$\frac{a}{c} = 1 : \frac{z_3 - z_1}{z_3 - z_2}.$$

Therefore the Möbius transformation carrying z_1, z_2, z_3 into $0, \infty, 1$ is

$$w = \Lambda(z) = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}. \qquad (10.21)$$

Next we determine the more general Möbius transformation $w = L(z)$ satisfying

$$L(z_1) = w_1, \quad L(z_2) = w_2, \quad L(z_3) = w_3,$$

where w_1, w_2, w_3 are three arbitrary (but distinct) finite points. As we have just seen, the transformation

$$\Lambda_1(z) = \frac{z - w_1}{z - w_2} : \frac{w_3 - w_1}{w_3 - w_2} \qquad (10.22)$$

[9] By $x:y$ is meant the *ratio of x to y*, i.e., the quantity x/y.

carries the points w_1, w_2, w_3 into the points $0, \infty, 1$. Therefore the transformation $\Lambda_1 L$ carries the points z_1, z_2, z_3 into the points $0, \infty, 1$, so that

$$\Lambda_1 L(z) = \Lambda(z) = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}. \tag{10.23}$$

Multiplying both sides of the equation

$$\Lambda_1 L(z) = \Lambda(z) \tag{10.24}$$

by Λ_1^{-1} from the left, we obtain

$$L(z) = \Lambda_1^{-1}\Lambda(z).$$

This solves the problem, since the functions $\Lambda(z)$ and $\Lambda_1(z)$, and hence $\Lambda_1^{-1}(z)$, are known [cf. (10.23) and (10.22)]. However, it is more convenient to use (10.24) directly, after writing $w = L(z)$. The result is

$$\Lambda_1(w) = \Lambda(z)$$

or

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}, \tag{10.25}$$

which expresses the Möbius transformation $w = L(z)$ in implicit form.

Remark. In finding the Möbius transformation carrying the points z_1, z_2, z_3 into the points w_1, w_2, w_3, it was assumed that all six points are finite. However, the case of infinite points is easily handled. For example, the transformation carrying the points ∞, z_2, z_3 into the points $0, \infty, 1$ has the form

$$w = \Lambda(z) = \frac{1}{z - z_2} : \frac{1}{z_3 - z_2},$$

which can be found by inspection or by writing (10.21) in the form

$$w = \frac{\dfrac{z}{z_1} - 1}{z - z_2} : \frac{\dfrac{z_3}{z_1} - 1}{z_3 - z_2}$$

and then taking the limit as $z_1 \to \infty$. Therefore (10.25) is replaced by

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{1}{z_1 - z_2} : \frac{1}{z_3 - z_2},$$

where it is assumed that the points w_1, w_2, w_3 are finite. Similarly, the transformation carrying the points z_1, ∞, z_3 into the points $0, \infty, 1$ has the form

$$\Lambda(z) = (z - z_1) : (z_3 - z_1),$$

and hence we have

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = (z - z_1) : (z_3 - z_1),$$

instead of (10.25). Finally, the transformation carrying the points z_1, z_2, ∞ into the points $0, \infty, 1$ has the form

$$\Lambda(z) = \frac{z - z_1}{z - z_2},$$

and hence we have

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2},$$

instead of (10.25).

In just the same way, we have to replace the left-hand side of (10.25) by

$$\frac{1}{w - w_2} : \frac{1}{w_3 - w_2}, \quad (w - w_1) : (w_3 - w_1) \quad \text{or} \quad \frac{w - w_1}{w - w_2},$$

depending on whether $w_1 = \infty$, $w_2 = \infty$ or $w_3 = \infty$. As a result, we arrive at the following mnemonic rule: If $z_k = \infty$ or $w_l = \infty$ $(k, l = 1, 2, 3)$, the differences involving z_k or w_l have to be replaced by 1. The reader can easily verify this rule by taking the appropriate limits (as $z_k \to \infty$ or $w_l \to \infty$) in equation (10.25).

Equation (10.25) implies an important general property of Möbius transformations. If a, b, c and d are arbitrary distinct finite complex numbers, then the ratio

$$\frac{c - a}{c - b} : \frac{d - a}{d - b},$$

denoted by (a, b, c, d), is called the *cross ratio* (or *anharmonic ratio*) of the four numbers (or points) a, b, c and d. If one of the four points a, b, c, d is the point at infinity, we define the cross ratio as the limit of the cross ratio of four finite points, three of which coincide with the three given finite points, as the fourth point approaches infinity. Thus, according to this definition, we have

$$\begin{aligned}
(\infty, b, c, d) &= \frac{1}{c - b} : \frac{1}{d - b},\\
(a, \infty, c, d) &= (c - a) : (d - a),\\
(a, b, \infty, d) &= 1 : \frac{d - a}{d - b},\\
(a, b, c, \infty) &= \frac{c - a}{c - b}.
\end{aligned}$$

Now let $w = L(z)$ be an arbitrary Möbius transformation, and let A, B, C, D be the points into which $L(z)$ maps four arbitrary (but distinct) points a, b, c, d. Since the points A, B and D are the images of the points a, b and d, it follows from (10.25) that the relation between z and $w = L(z)$ is given by

$$\frac{w - A}{w - B} : \frac{D - A}{D - B} = \frac{z - a}{z - b} : \frac{d - a}{d - b},$$

where differences involving the point at infinity have to be replaced by 1. Moreover, since the point C is the image of the point c, we have

$$\frac{C-A}{C-B}:\frac{D-A}{D-B}=\frac{c-a}{c-b}:\frac{d-a}{d-b},$$

(where again differences involving the point at infinity have to be replaced by 1), or equivalently

$$(A, B, C, D) = (a, b, c, d).$$

In other words, *the cross ratio of any four distinct points is invariant under a Möbius transformation.*

47. Mapping of a Circle onto a Circle[10]

Using the circle-preserving property of Möbius transformations and the possibility of mapping any given triple of distinct points z_1, z_2, z_3 into any other given triple of distinct points w_1, w_2, w_3, we obtain the following basic result:

THEOREM 10.7. *Let γ and Γ be any two straight lines or circles, and let z_1, z_2, z_3 and w_1, w_2, w_3 be any two triples of distinct points belonging to γ and Γ, respectively. Then there exists a Möbius transformation $w = L(z)$ mapping γ onto Γ in such a way that*

$$L(z_k) = w_k \qquad (k = 1, 2, 3). \tag{10.26}$$

Proof. Construct the Möbius transformation $w = L(z)$ satisfying the conditions (10.26), which according to Theorem 10.6 and the subsequent construction, exists and is unique. According to Theorem 10.4, $w = L(z)$ maps the straight line or circle γ onto another straight line or circle Γ^*. But since γ goes through the points z_1, z_2 and z_3, Γ^* must go through the points w_1, w_2 and w_3. Moreover, since two different straight lines or circles cannot be drawn through the same three points, Γ^* must coincide with Γ, as asserted.

Remark. Again consider two arbitrary straight lines or circles γ and Γ (which may coincide). Let G be one of the two domains with boundary γ, and let $\mathscr{G}$ be one of the two domains with boundary Γ, so that G is either a half-plane, the interior of a circle or the exterior of a circle, and the same is true of $\mathscr{G}$. We now show how to map G onto $\mathscr{G}$. Choose an arbitrary triple of distinct points z_1, z_2, z_3 on γ, and suppose an observer moving along γ in the direction from z_1 to z_3 through z_2 finds the domain G on his *left*, say. Next choose a triple of distinct points w_1, w_2, w_3 on Γ such that an observer moving along Γ in the direction from w_1 to w_3 through w_2 finds the domain $\mathscr{G}$ on his left, but let w_1, w_2, w_3 be otherwise arbitrary. As in Theorem 10.7, we form the Möbius transformation $w = L(z)$ which satisfies the conditions (10.26) and hence maps γ onto Γ. Then $w = L(z)$ also maps G onto $\mathscr{G}$, i.e., $\mathscr{G} = L(G)$. In fact, if δ is a segment of the normal to the curve γ drawn from the point z_2 and pointing into the interior of G, so that an observer at z_2 facing in the direction established on γ finds δ on his left, then, since the mapping $w = L(z)$ is conformal, an observer at w_2 facing in the direction established on Γ will also find the image $\Delta = L(\delta)$, which is a line segment or circular arc, on his left (see Figure 10.2 and Remark 2, p. 156). Therefore $\Delta \subset \mathscr{G}$, and hence $\mathscr{G}$ contains images of certain points belonging to G (i.e., the points of the segment δ). But, according to the remark on p. 170, $L(G)$ is one of the two domains with boundary $\Gamma = L(\gamma)$, in fact just the domain containing the image of any point in G. In other words, $\mathscr{G} = L(G)$, as asserted.

[10] As usual, a straight line is regarded as a limiting case of a circle (cf. p. 168).

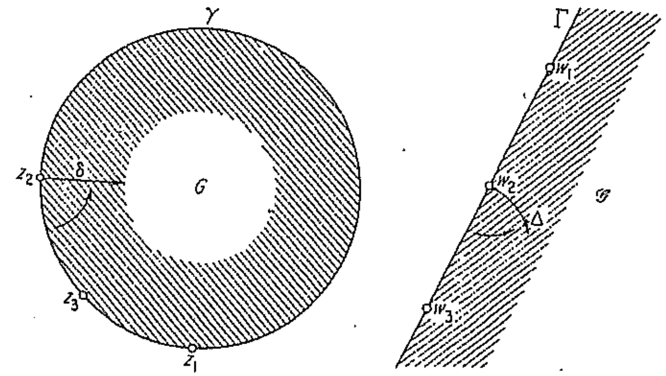

FIGURE 10.2

Example. Find a conformal mapping of the upper half-plane $\operatorname{Im} z > 0$ *onto the interior of the unit circle.*

To solve this problem, we choose $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, say, so that the upper half-plane is on the left of an observer moving along the real axis in the direction from z_1 to z_3 through z_2. We also choose three points w_1, w_2, w_3 on the unit circle, such that the interior of the circle is on the left of an observer moving along the circle in the direction from w_1 to w_3 through w_2. For simplicity, we choose $w_1 = 1$, $w_2 = i$, $w_3 = -1$. Then the desired Möbius transformation satisfies the conditions $L(z_k) = w_k$, $k = 1, 2, 3$, and can be represented in the form

$$\frac{w-1}{w-i}:\frac{-1-1}{-1-i}=\frac{z+1}{z}:\frac{1+1}{1}$$

or

$$w = \frac{z-i}{iz-1},$$

where we have used (10.25).

48. Symmetry Transformations

Let z_1 and z_2 be two points which are symmetric with respect to a given straight line γ, i.e., such that γ is the perpendicular bisector of the line segment joining z_1 and z_2. By definition, the straight line passing through z_1 and z_2 is orthogonal to γ. Moreover, the center of any circle δ passing through z_1 and z_2 lies on γ, and hence δ is also orthogonal to γ. It is easy to see that the converse is true as well, i.e., if every straight line or circle passing through a pair of points z_1 and z_2 is orthogonal to a given straight line γ, then z_1 and z_2 are symmetric with respect to γ. Generalizing the concept of symmetry with respect to a straight line, we introduce the following definition: *Two points z_1 and z_2 are symmetric with respect to a given circle γ if and only if every straight line or circle passing through z_1 and z_2 is orthogonal to γ.*

THEOREM 10.8. *Let z_1 and z_2 be any two points symmetric with respect to a given straight line or circle γ, and let $w = L(z)$ be any Möbius transformation. Then the points $w_1 = L(z_1)$ and $w_2 = L(z_2)$ are symmetric with respect to the straight line or circle $\Gamma = L(\gamma)$.*[11]

Proof. We have to show that an arbitrary straight line or circle Δ passing through w_1 and w_2 is orthogonal to Γ. Let $z = L^{-1}(w)$ be the inverse of the transformation $w = L(z)$. Clearly, L^{-1} is also a Möbius transformation, and

$$L^{-1}(w_1) = z_1, \quad L^{-1}(w_2) = z_2, \quad L^{-1}(\Gamma) = \gamma.$$

Moreover, $\delta = L^{-1}(\Delta)$ is a straight line or circle passing through z_1 and z_2. Since z_1 and z_2 are symmetric with respect to γ, by hypothesis, it follows that δ is orthogonal to γ. But then, since the mapping $w = L(z)$ is conformal (see Sec. 33), $\Delta = L(\delta)$ is orthogonal to Γ, and the proof is complete.

COROLLARY. *There is only one point z_2 symmetric to a given point z_1 with respect to a given straight line or circle γ.*

Proof. If γ is a straight line, the statement is obvious. Thus let γ be a circle, and suppose that besides z_2, there is another point $z_2' \neq z_2$ symmetric to z_1 with respect to γ. Choosing a Möbius transformation $w = L(z)$ mapping γ onto a straight line Γ, we find that $w_2 = L(z_2)$ and $w_2' = L(z_2')$ are two distinct points symmetric with respect to Γ, which is impossible.

Remark. Suppose $w = L(z)$ maps a straight line or circle γ onto a circle Γ with center w_1, and let z_1 be the inverse image of w_1. Then the point z_2 symmetric to z_1 with respect to γ must be mapped into the point at infinity. To see this, we note that $w_2 = \infty$ is symmetric to $w_1 = 0$ with respect to the circle Γ; since any straight line or circle passing through 0 and ∞, i.e., any straight line passing through the center of Γ, is orthogonal to Γ. The uniqueness of w_2 follows from the corollary.

[11] It is in this sense that Möbius transformations are said to be *symmetry-preserving*.

Let γ be an arbitrary straight line or circle. A transformation of the extended plane into itself, which carries each point z into the point z^* symmetric to z with respect to γ is called a *symmetry transformation* with respect to γ or a *reflection* in γ. In the case where γ is a circle, the transformation is also called an *inversion* in γ. We now derive analytical expressions for symmetry transformations.

First let γ be a straight line with an assigned direction, and consider reflection in γ. The straight line γ is completely characterized by one of its points a and by the unit vector

$$e^{i\theta} = \cos\theta + i\sin\theta$$

pointing in the direction of γ. Suppose we carry out the entire linear transformation

$$z = L(w) = a + e^{i\theta}w, \tag{10.27}$$

which obviously maps the real axis onto γ, since (10.27) corresponds to a shift by the vector a (carrying the origin of coordinates to the point a), followed by a rotation through the angle θ about the point a. Since the inverse transformation $w = L^{-1}(z)$ maps γ onto the real axis, it maps every pair of points z and z^* symmetric with respect to γ into a pair of points w and w^* symmetric with respect to the real axis. But the points w and w^* are represented by two conjugate complex numbers, i.e.,

$$w = t, \qquad w^* = \bar{t}. \tag{10.28}$$

Therefore $z - a = e^{i\theta}t$, and

$$\overline{z-a} = e^{-i\theta}\bar{t}, \qquad z^* - a = e^{i\theta}w^* = e^{i\theta}\bar{t}. \tag{10.29}$$

Eliminating $\bar{t}$ from (10.29), we obtain

$$z^* - a = e^{2i\theta}(\overline{z-a}). \tag{10.30}$$

According to (10.30), reflection in a straight line γ going through a point a at an angle θ with the real axis can be accomplished by first constructing the vector $\overline{z-a}$ which is the reflection of the vector $z - a$ in the real axis, and then rotating $\overline{z-a}$ through the angle 2θ about the point a.

Next let γ be a circle, and consider inversion in γ. Let R $(0 < R < \infty)$ be the radius and a the center of γ. We begin by finding a Möbius transformation which maps γ onto the real axis. The simplest approach is to choose the transformation

$$z = L(w) = a + R\frac{1 + iw}{1 - iw},$$

which maps the three points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$ of the real axis into the points $z_1 = a - iR$, $z_2 = a + R$, $z_3 = a + iR$ of the circle γ. The inverse transformation $w = L^{-1}(z)$ maps γ onto the real axis, and maps every pair of points z and z^* symmetric with respect to γ into a pair of points w and w^* symmetric with respect to the real axis. As before, the points w and w^* are represented by two conjugate complex numbers (10.28). Therefore

$$z - a = R\frac{1 + it}{1 - it},$$

and

$$\overline{z - a} = R\frac{1 - i\bar{t}}{1 + i\bar{t}}, \qquad z^* - a = R\frac{1 + i\bar{t}}{1 - i\bar{t}}. \tag{10.31}$$

Multiplying the two equations (10.31), we obtain

$$(\overline{z - a})(z^* - a) = R^2$$

or

$$z^* - a = \frac{R^2}{\overline{z - a}}. \tag{10.32}$$

In particular, it follows from (10.32) that

$$\text{Arg}\,(z^* - a) = -\text{Arg}\,(\overline{z - a}) = \text{Arg}\,(z - a)$$

and

$$|z - a|\,|z^* - a| = R^2.$$

Therefore the points z and z^* lie on the same ray emanating from the center of γ, and the product of their distances from the center of γ equals the square of the radius of γ. These two conditions, equivalent to formula (10.32), determine the position of one of the points z, z^* with respect to the other, and completely characterize the operation of inversion in the circle γ, with equation $|z - a| = R$.

Remark. We note that any symmetry transformation reduces to consecutive application of a linear transformation (entire or fractional), followed by reflection in the real axis. In fact, according to (10.30), reflection in a straight line can be represented in the form

$$z_1 = \bar{a} + e^{-2i\theta}(z - a), \qquad z^* = \bar{z}_1,$$

while, according to (10.32), reflection in a circle can be represented in the form

$$z_1 = \bar{a} + \frac{R^2}{z - a}, \qquad z^* = \bar{z}_1.$$

Since any Möbius transformation is conformal and circle-preserving, and since reflection in the real axis has the same properties, except that while preserving the magnitudes of angles it reverses the directions in which they are measured, we see that the most general symmetry transformation is a conformal mapping of the second kind (see p. 120) which is also circle-preserving.

49. Examples

We now give two examples illustrating the use of the symmetry-preserving property of Möbius transformations.

Example 1. *Find a conformal mapping of the upper half-plane* Π_U: Im $z > 0$ *onto the disk* K: $|w| < R$, *such that a given point* $\alpha \in \Pi_U$ *is mapped into the center of* K.[12]

Any such mapping

$$w = L(z) = \frac{az + b}{cz + d},$$

provided it exists, vanishes for $z = \alpha$, and hence α is the zero of $L(z)$. But the point $\bar{\alpha}$ symmetric to α with respect to the real axis must be mapped into the point symmetric to the center $w = 0$ of K with respect to the boundary of K, i.e., the circle C: $|w| = R$. Therefore $\bar{\alpha}$ must be mapped into the point at infinity (see the remark on p. 178), so that $L(\bar{\alpha}) = \infty$ and $\bar{\alpha}$ is the pole of $L(z)$. It follows that $L(z)$ has the form

$$w = L(z) = \frac{a(z - \alpha)}{c(z - \bar{\alpha})} = \lambda\frac{z - \alpha}{z - \bar{\alpha}}, \tag{10.33}$$

where λ is a nonzero complex number.

We now show that (10.33) maps Π_U onto K, with $z = \alpha$ going into $w = 0$, if we choose $|\lambda| = R$. Since $L(\alpha) = 0$ for any λ, by construction, it is sufficient to show that (10.33) maps the real axis onto the circle C. If $z = x$ is an arbitrary real number, then $x - \alpha$ and $x - \bar{\alpha}$ are complex conjugates, and hence

$$|w| = |L(x)| = \left|\lambda\frac{x - \alpha}{x - \bar{\alpha}}\right| = |\lambda|\left|\frac{x - \alpha}{x - \bar{\alpha}}\right| = |\lambda| = R.$$

Therefore (10.33) maps the real axis *into* C. But since any three distinct points of the real axis are mapped into three distinct points of C, it follows from Theorem 10.7 that the real axis is mapped *onto* C.

In (10.33) the argument of λ is left unspecified. The geometric reason for this indeterminacy is clear: Going from one value of λ to another in (10.33),

[12] In expressions like Π_U: Im $z > 0$ and K: $|w| < R$, the colon means "with equation."

while keeping $|\lambda| = R$ fixed, is equivalent to changing the arguments of all points w by the same quantity, i.e., to rotating the disk K about its center $w = 0$. Such a rotation transforms K into itself while leaving its center fixed, and hence does not violate the conditions of the problem. Thus, if the problem is to have a unique solution, we must impose an extra condition on $L(z)$. For example, we might require either that

1. *A given point $x = x_0$ of the real axis should go into the point $w = R$ of the circle C,* or that
2. *The derivative $L'(\alpha)$ should be a positive real number.* (Geometrically, this means that the mapping does not change the slopes of tangents to curves passing through the point α.)

Imposing condition 1, we find from (10.33) that

$$R = L(x_0) = \lambda \frac{x_0 - \alpha}{x_0 - \bar{\alpha}},$$

so that

$$\lambda = R \frac{x_0 - \bar{\alpha}}{x_0 - \alpha},$$

and hence

$$L(z) = R \frac{x_0 - \bar{\alpha}}{x_0 - \alpha} \frac{z - \alpha}{z - \bar{\alpha}}.$$

Moreover, we still have

$$|\lambda| = R \left| \frac{x_0 - \bar{\alpha}}{x_0 - \alpha} \right| = R,$$

as required. Imposing condition 2, with $\alpha = \xi + i\eta$ $(\eta > 0)$, we have

$$L'(\alpha) = \frac{\lambda}{\alpha - \bar{\alpha}} = \frac{\lambda}{2\eta i},$$

so that λ/i is a positive real number. But, on the other hand, $|\lambda|$ must equal R. Therefore $\lambda = iR$, and

$$L(z) = iR \frac{z - \alpha}{z - \bar{\alpha}}.$$

Example 2. Find a conformal mapping of the disk K: $|z| < R$ onto itself such that a given point $\alpha \in K$ is mapped into the center of K.

Any such mapping

$$w = L(z) = \frac{az + b}{cz + d},$$

provided it exists, vanishes for $z = \alpha$, and hence α is the zero of $L(z)$. But the point α^* symmetric to α with respect to the boundary of K, i.e., the circle C: $|z| = R$, must be mapped into a point symmetric to the center $w = 0$ of K

with respect to C. Therefore α^* must be mapped into the point at infinity, so that $L(\alpha^*) = \infty$ and α^* is the pole of $L(z)$. It follows that $L(z)$ has the form

$$w = L(z) = \lambda \frac{z - \alpha}{z - \alpha^*} \tag{10.34}$$

[cf. (10.33)], where λ is a nonzero complex number. According to (10.32), the point α^* symmetric to α with respect to C is

$$\alpha^* = \frac{R^2}{\bar{\alpha}},$$

and hence (10.34) becomes

$$w = L(z) = -\lambda\bar{\alpha} \frac{z - \alpha}{R^2 - \bar{\alpha}z} = \mu \frac{z - \alpha}{R^2 - \bar{\alpha}z}. \tag{10.35}$$

We now show that (10.35) maps K onto K, with $z = \alpha$ going into $w = 0$, if we choose $|\mu| = R^2$. Since $L(\alpha) = 0$ for any μ, by construction, it is sufficient to show that (10.35) maps the circle C: $|z| = R$ onto itself. If

$$z = Re^{i\theta} \qquad (0 \leqslant \theta < 2\pi)$$

is an arbitrary point of C, then

$$w = L(Re^{i\theta}) = \mu \frac{re^{i\theta} - \alpha}{R^2 - \bar{\alpha} Re^{i\theta}} = \frac{\mu}{Re^{i\theta}} \frac{Re^{i\theta} - \alpha}{Re^{-i\theta} - \bar{\alpha}},$$

and hence

$$|w| = |L(Re^{i\theta})| = \left| \frac{\mu}{Re^{i\theta}} \frac{Re^{i\theta} - \alpha}{Re^{-i\theta} - \bar{\alpha}} \right| = \frac{|\mu|}{R} = \frac{R^2}{R} = R.$$

Therefore (10.35) maps C *into* C, and hence *onto* C, by the same argument as before. This example is considered further in Problems 10.18–10.22.

*50. Lobachevskian Geometry

In the preceding section, we saw that every function of the form

$$w = L(z) = \mu \frac{z - \alpha}{R^2 - \bar{\alpha}z} \qquad (|\alpha| < R, \quad |\mu| = R^2) \tag{10.36}$$

transforms the disk K: $|z| < R$ into itself. In particular, the function

$$\bar{\mu} \frac{z - \bar{\alpha}}{R^2 - \alpha z},$$

which we denote by $\bar{L}(z)$, has the same property, and obviously $\overline{L(z)} = \bar{L}(\bar{z})$. Moreover, it is easy to see that by letting α and μ in (10.36) take all possible