its vertex at the origin is enlarged n times. Let l be a curve in the ζ-plane, with equation $\zeta = \lambda(t)$, $t \in [a, b]$, which goes through the point $\zeta = 0$ and has a tangent τ with inclination θ at $\zeta = 0$. Then, by a slight modification of the previous argument, we find that the curve L in the η-plane with equation $\eta = P_n^*[\lambda(t)] = \Lambda(t)$ goes through the point $\eta = 0$ and has a tangent with inclination

$$\Theta = n\theta - \operatorname{Arg} a_n$$

at $\eta = 0$. The rest of the proof follows as before (note that $w = 1/\eta$).

37. The Mapping $w = (z - a)^n$

We now make a detailed study of the mapping

$$w = (z - a)^n \qquad (n > 1). \tag{9.8}$$

This function maps the extended z-plane onto the extended w-plane in such a way that every point w has n distinct inverse images, with the exception of the two points $w = 0$ and $w = \infty$, for which the n inverse images "coalesce" into the single points $z = a$ and $z = \infty$, respectively. To find the n inverse images of w when $w \neq 0$, $w \neq \infty$, we solve (9.8) for z, obtaining

$$z = a + \sqrt[n]{w} = a + \sqrt[n]{|w|}\left(\cos\frac{\operatorname{Arg} w}{n} + i\sin\frac{\operatorname{Arg} w}{n}\right). \tag{9.9}$$

Obviously, the n distinct points (9.9) lie at the vertices of a regular n-gon with its center at the point $z = a$. The mapping (9.8) is conformal at all points except $z = a$, $z = \infty$, and every angle with its vertex at one of these two points is enlarged n times.

To get a clearer picture of the mapping (9.8), we observe that

$$|w| = |z - a|^n, \qquad \operatorname{Arg} w = n \operatorname{Arg}(z - a),$$

which implies that every circle of radius r with its center at the point $z = a$ is mapped into a circle of radius r^n with its center at the point $w = 0$. Moreover, as the point z goes around the circle $|z - a| = r$ once in the positive direction [so that $\operatorname{Arg}(z - a)$ increases continuously by 2π], the image point w goes around the circle $|w| = r^n$ n times in the same direction [since Arg w increases continuously by $2n\pi$]. We also note that as the point z sweeps out the ray

$$\operatorname{Arg}(z - z) = \varphi_0 + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

going from a to ∞, the image point w sweeps out the ray

$$\operatorname{Arg} w = n\varphi_0 + 2m\pi \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

going from 0 to ∞.

Next consider the domain G consisting of all points z such that

$$\varphi_0 + 2k\pi < \operatorname{Arg}(z - a) < \varphi_1 + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

where $0 < \varphi_1 - \varphi_0 \leqslant 2\pi/n$. Such a domain will be called the *interior* of the angle of $\varphi_1 - \varphi_0$ radians formed by the rays

$$\operatorname{Arg}(z - a) = \varphi_0 + 2k\pi, \quad \operatorname{Arg}(z - a) = \varphi_1 + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots),$$

the term *angle* itself being reserved either for the figure formed by these two rays (an "unbounded curve") or for the quantity $\varphi_1 - \varphi_0$. Then the image of G under the mapping (9.8) is the domain

$$n\varphi_0 + 2m\pi < \operatorname{Arg} w < n\varphi_1 + 2m\pi \qquad (m = 0, \pm 1, \pm 2, \ldots),$$

i.e., the interior of the angle of $n(\varphi_1 - \varphi_0)$ radians with its vertex at the origin of the w-plane (see Figure 9.1). Not only is the function $w = (z - a)^n$

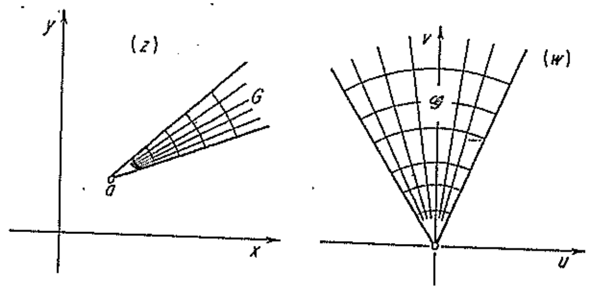

FIGURE 9.1

conformal on G, as already noted, but it is also one-to-one on G. In fact, since $w = (z - a)^n$ is single-valued, we need only verify that every point w has only one inverse image in G. Since the n inverse images of the point w lie at the vertices of a regular n-gon in the z-plane with center at a, two inverse images can belong to the interior of the same angle with vertex at a only if the angle exceeds $2\pi/n$. But $0 < \varphi_1 - \varphi_0 \leqslant 2\pi/n$ by hypothesis, and hence every point of $\mathscr{G}$ has only one inverse image in G, as asserted. Thus, *the function $w = (z - a)^n$ is a one-to-one conformal mapping of the interior of one angle onto the interior of another angle which is n times larger.*

Of course, it would be quite incorrect to conclude that the function $w = (z - a)^n$ maps every straight line into a straight line, and every circle into a circle. For example, suppose $a = 0$, $n = 2$, so that $w = (z - a)^n$ reduces to $w = z^2$, and consider the effect of this mapping on straight lines parallel to the coordinate axes. Every line parallel to the imaginary axis (except the axis itself) has an equation of the form

$$z = b + it \qquad (-\infty < t < \infty),$$

where $b \neq 0$ is a real number, and the image of this line under the mapping $w = z^2$ has the equation

$$w = (b + it)^2. \tag{9.10}$$

Separating real and imaginary parts of (9.10), we obtain the corresponding parametric equations

$$u = b^2 - t^2, \qquad v = 2bt,$$

where u and v are the rectangular coordinates of the point w. Elimination of the parameter t leads to

$$v^2 = 4b^2(b^2 - u), \tag{9.11}$$

which is the equation of a parabola opening to the left, with axis lying along the real axis and focus at the origin. Similarly, every line parallel to the real axis (except the axis itself) has an equation of the form

$$z = t + ic \qquad (-\infty < t < \infty),$$

where $c \neq 0$ is a real number, and it is easily verified that the image of this line under the mapping $w = z^2$ is the parabola

$$v^2 = 4c^2(c^2 + u) \tag{9.12}$$

opening to the right, with axis lying along the real axis and focus at the origin. Thus, *the mapping $w = z^2$ transforms the two one-parameter families of straight lines parallel to (but distinct from) the coordinate axes into two one-parameter families of parabolas* (9.11) *and* (9.12), *with axes lying along the real axis and a common focus at the origin* (see Figure 9.2). Moreover, since the two families of straight lines in the z-plane form an *orthogonal system* (i.e., every curve in one family is orthogonal to every curve in the other family, and *vice versa*), and since the mapping $w = z^2$ is conformal (except at $z = 0$), it follows that the two families of parabolas in the w-plane also form an orthogonal system.[5]

Remark. It should be kept in mind that $w = z^2$ is not a one-to-one mapping, and in fact, every point in the w-plane except $w = 0$ and $w = \infty$ has two inverse images. In particular, the inverse image of the parabola (9.11) consists of the two straight lines $z = b + it$ and $z = -b + it$ which are symmetric with respect to the imaginary axis, while the inverse image of the parabola (9.12) consists of the two straight lines $z = t + ic$ and $z = t - ic$ which are symmetric with respect to the real axis. However, if we confine ourselves to a half-plane G whose boundary consists of a line passing through the origin (i.e., the interior of an angle of π radians with vertex at the origin),

[5] The x and y-axes themselves are orthogonal, but their images, i.e., the line segments $u \geq 0,\ v = 0$ and $u \leq 0,\ v = 0$ (each traversed twice) are not orthogonal, since the derivative of $w = z^2$ vanishes at $z = 0$ (cf. Theorem 9.2).

then the correspondence between G and its image $\mathscr{G}$ under $w = z^2$ is one-to-one. In fact, $\mathscr{G}$ is the interior of an angle of 2π radians with vertex at the origin, both of whose sides coalesce to form a single ray emanating from the origin.

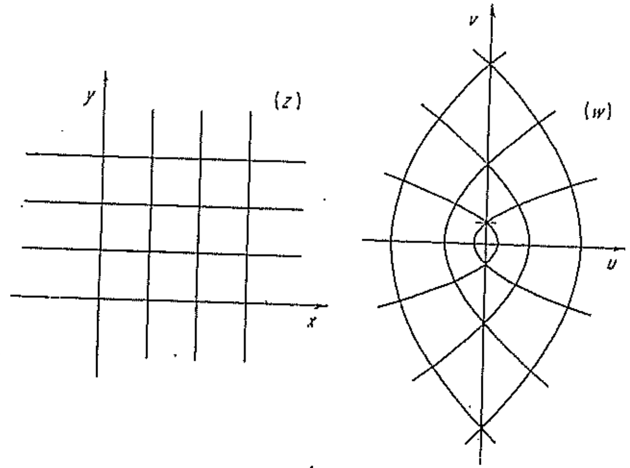

FIGURE 9.2

38. The Exponential

An entire function which is not a polynomial is called an *entire transcendental function*. The simplest example of such a function is the *exponential (function)* e^z or $\exp z$, obtained by suitably extending the familiar function e^x, defined for the real variable x, to the case where x takes arbitrary complex values, i.e., is replaced by the complex variable $z = x + iy$. It is not hard to show that the real exponential $f(x) = e^x$ is the unique function with the following properties:

1. $f(x)$ is defined and single-valued for all real x, takes only real values, and in particular takes the value e when $x = 1$;
2. $f(x)$ satisfies the *addition theorem*

$$f(x_1 + x_2) = f(x_1)f(x_2);$$

3. $f(x)$ is continuous for all x.[6]

The complex function $f(z) = e^z$ can be characterized in much the same way:

[6] This result follows at once by a specialization of the proof of Theorem 9.3.