8

GEOMETRIC INTERPRETATION OF THE DERIVATIVE. CONFORMAL MAPPING

31. Geometric Interpretation of $\operatorname{Arg} f'(z)$

Let l be a continuous curve with equation $z = \lambda(t)$, $t \in [a, b]$, and suppose $\lambda(t)$ is differentiable at a point $t_0 \in [a, b]$ (relative to the set $[a, b]$). Let $\{t_n\}$ be an arbitrary sequence of points in $[a, b]$ converging to t_0 ($t_n \neq t_0$), and consider the difference quotient

$$r_n = \frac{\lambda(t_n) - \lambda(t_0)}{t_n - t_0}. \tag{8.1}$$

Obviously $r_n \to r_0 = \lambda'(t_0)$ as $n \to \infty$.

DEFINITION. *The curve l is said to have a tangent at the point $z_0 = \lambda(t_0)$ if the limit*

$$\theta = \lim_{n \to \infty} \operatorname{Arg} r_n \tag{8.2}$$

exists, and then the tangent is said to have inclination θ. Geometrically, the tangent to l at z_0 is represented by the ray τ emanating from z_0 which makes the angle θ with the positive real axis.[1]

[1] Formula (8.2) means that given any $\varepsilon > 0$, there is an integer $N(\varepsilon) > 0$ and a sequence $\{\theta_n\}$, where each θ_n is a value of $\operatorname{Arg} r_n$, such that $|\theta_n - \theta| < \varepsilon$ for all $n > N(\varepsilon)$. Clearly, θ is only defined to within a multiple of 2π. The angle θ will always be measured from the positive real axis to the tangent τ (in the counterclockwise direction for a positive value of θ).

Remark 1. Clearly, if $\lambda'(t_0) \neq 0$, l has a tangent at z_0, since

$$\theta = \lim_{n \to \infty} \operatorname{Arg} r_n = \operatorname{Arg} r_0 = \operatorname{Arg} \lambda'(t_0)$$

(see p. 35), and then the inclination of the tangent is just the argument of the complex number $\lambda'(t_0)$. On the other hand, if $\lambda'(t_0) = 0$, l may or may not have a tangent at z_0, since the fact that $r_n \to 0$ implies only that $|r_n| \to 0$ and says nothing about the behavior of $\operatorname{Arg} r_n$ (see p. 33). However, if l has a tangent at z_0, then $\lambda(t_n) \neq \lambda(t_0)$ for all t_n sufficiently close to t_0, since Arg 0 is meaningless.[2]

Remark 2. As we have defined it, the tangent is a ray, not a vector. If $\lambda'(t_0) \neq 0$, we can also introduce a *tangent vector* to l at z_0, defined as the vector of length $|\lambda'(t_0)|$ which makes the angle θ with the positive real axis.

THEOREM 8.1. *Let G be a domain, and let $f(z)$ be a continuous function of a complex variable defined on G. Suppose $f(z)$ has a nonzero derivative $f'(z_0)$ at a point $z_0 \in G$, and let l be a curve which passes through z_0 and has a tangent τ at z_0. Then $w = f(z)$ maps l into a curve L in the w-plane which passes through the point $w_0 = f(z_0)$ and has a tangent T at w_0. Moreover, the inclination of T exceeds the inclination of τ by the angle* $\operatorname{Arg} f'(z_0)$.

Proof. Suppose l has the equation $z = \lambda(t)$, $t \in [a, b]$, and let $z_0 = \lambda(t_0)$. By hypothesis,

$$\theta = \lim_{n \to \infty} \operatorname{Arg} r_n$$

exists, where r_n is given by (8.1). The function $w = f(z)$ maps l into a curve L in the w-plane with equation

$$w = f[\lambda(t)] = \Lambda(t), \qquad t \in [a, b], \tag{8.3}$$

where $w_0 = f(z_0) = \Lambda(t_0)$. Let $\{t_n\}$ be an arbitrary sequence of points in $[a, b]$ converging to t_0, and let

$$R_n = \frac{\Lambda(t_n) - \Lambda(t_0)}{t_n - t_0}.$$

Then the tangent to L at w_0 has inclination

$$\Theta = \lim_{n \to \infty} \operatorname{Arg} R_n,$$

provided this limit exists. Clearly we have

$$R_n = \frac{\Lambda(t_n) - \Lambda(t_0)}{\lambda(t_n) - \lambda(t_0)} \frac{\lambda(t_n) - \lambda(t_0)}{t_n - t_0},$$

[2] This condition is automatically satisfied if $\lambda'(t_0) \neq 0$, since otherwise $r_n = 0$ for t arbitrarily close to t_0, which implies that $\lambda'(t_0) = 0$, contrary to hypothesis.

where the first factor in the right-hand side is well defined, since $\lambda(t) \neq \lambda(t_0)$ for all t_n sufficiently close to t_0 (l is assumed to have a tangent at z_0). Therefore

$$\begin{aligned}\Theta = \lim_{n\to\infty} \operatorname{Arg} R_n &= \lim_{n\to\infty} \operatorname{Arg} \frac{\Lambda(t_n) - \Lambda(t_0)}{\lambda(t_n) - \lambda(t_0)} \frac{\lambda(t_n) - \lambda(t_0)}{t_n - t_0} \\ &= \lim_{n\to\infty} \operatorname{Arg} \frac{w_n - w_0}{z_n - z_0} r_n = \lim_{n\to\infty} \operatorname{Arg} \frac{w_n - w_0}{z_n - z_0} + \lim_{n\to\infty} \operatorname{Arg} r_n \qquad (8.4) \\ &= \operatorname{Arg} \lim_{n\to\infty} \frac{w_n - w_0}{z_n - z_0} + \theta = \operatorname{Arg} f'(z_0) + \theta,\end{aligned}$$

where $w_n = \Lambda(t_n)$, $z_n = \lambda(t_n)$ and θ is the inclination of τ at z_0.[3] It follows from (8.4) that Θ exists and that

$$\Theta - \theta = \operatorname{Arg} f'(z_0),$$

as asserted. We note that things are particularly simple in the case where $\lambda'(t_0) \neq 0$, since then

$$\begin{aligned}\Theta &= \operatorname{Arg} \Lambda'(t_0) = \operatorname{Arg} [f'(z_0)\lambda'(t_0)] \\ &= \operatorname{Arg} f'(z_0) + \operatorname{Arg} \lambda'(t_0) = \operatorname{Arg} f'(z_0) + \theta,\end{aligned}$$

by the rule for differentiating the composite function (8.3).

Now let l_1 and l_2 be two curves with a common initial point z_0, which have tangents τ_1 and τ_2 at z_0, and suppose the angle between τ_1 and τ_2 is measured from τ_1 to τ_2. Suppose l_1 and l_2 have images L_1 and L_2 under $f(z)$. Then, according to Theorem 8.1, if $f'(z_0) \neq 0$, L_1 and L_2 have tangents T_1 and T_2 at the point $w_0 = f(z_0)$, where T_1 and T_2 are obtained by rotating τ_1 and τ_2 through the same angle $\operatorname{Arg} f'(z_0)$. Therefore the angle between L_1 and L_2 equals the angle between l_1 and l_2, and is measured in the same direction, i.e., from L_1 to L_2. In other words, a continuous function $w = f(z)$ with a nonzero derivative $f'(z_0)$ maps all curves in the z-plane which pass through z_0 and have tangents at z_0 into curves in the w-plane which pass through $w_0 = f(z_0)$ and have tangents at w_0, and, moreover, *the mapping preserves angles between curves.* A mapping by a continuous function which preserves angles between curves passing through a given point z_0 is said to be *conformal at* z_0 (cf. p. 88). If a conformal mapping preserves the directions in which angles are measured (as well as their magnitudes), it is called a *conformal mapping of the first kind*, but if it reverses the directions in which angles are measured, it is called a *conformal mapping of the second kind.* If a mapping is conformal at all points of a domain G, it is said to be *conformal on* G. Thus Theorem 8.1 has the following consequence:

[3] In reversing the order of the operations Arg and $\lim\limits_{n\to\infty}$, we have used the fact that $f'(z_0) \neq 0$.

THEOREM 8.2. *Let G be a domain, and let $f(z)$ be an analytic function on G. Then $f(z)$ is a conformal mapping of the first kind at every point of G where $f'(z) \neq 0$.*

Example 1. Reflection in the real axis, i.e., the transformation $w = \bar{z}$, is a conformal mapping of the second kind. A more general example is the complex conjugate

$$w = \overline{f(z)}$$

of an analytic function $f(z)$, where $f'(z) \neq 0$.

Example 2. At a point where the derivative vanishes, angles may or may not be preserved, as can be seen by comparing the mappings

$$\begin{aligned}f_1(z) &= r^2(\cos \Phi + i \sin \Phi) = rz, \\ f_2(z) &= r^2(\cos 2\Phi + i \sin 2\Phi) = z^2\end{aligned}$$

at the point $z = 0$.

32. Geometric Interpretation of $|f'(z)|$

As we have just seen, $\operatorname{Arg} f'(z_0)$ represents the rotation undergone by the tangent to a curve l at the point $z_0 \in l$ when transforming to the new curve $L = f(l)$ and the new point $w_0 = f(z_0)$. In particular, if $f'(z_0)$ is a positive real number, the tangents to l at z_0 and to L at w_0 are parallel and point in the same direction.

To explain the geometric meaning of the quantity $|f'(z_0)|$, i.e., the absolute value of the derivative at z_0, we note that

$$|f'(z_0)| = \lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

The numbers $|z - z_0|$ and $|f(z) - f(z_0)|$ are the distance between the points z and z_0 in the z-plane, and the distance between their images $f(z)$ and $f(z_0)$ in the w-plane, respectively. Thus, interpreting

$$\frac{|f(z) - f(z_0)|}{|z - z_0|}$$

as the *linear magnification ratio* (or simply the *magnification*) of the vector $z - z_0$ under the mapping $w = f(z)$,[4] we can regard $|f'(z_0)|$ as the *magnification at the point* z_0 under $w = f(z)$.

[4] Here the word *magnification* is used in a general sense, and can correspond to *stretching* if $|f'(z_0)| > 1$ or *shrinking* if $|f'(z_0)| < 1$ [or neither if $|f'(z_0)| = 1$].

Remark. The size of the magnification at the point z_0 does not depend on the choice of the finite vector $z - z_0$ drawn from z_0, since $|f'(z_0)|$ is not the actual magnification of any such vector, but rather the limiting magnification as $z \to z_0$.

33. The Mapping $w = \dfrac{az + b}{cz + d}$

To illustrate the above considerations, we now examine the *fractional linear transformation* or *Möbius transformation*

$$L(z) = \frac{az + b}{cz + d}, \tag{8.5}$$

where a, b, c, d are arbitrary complex numbers (except that c and d are not both zero). First suppose that $c = 0$. Then $L(z)$ reduces to

$$L(z) = \alpha z + \beta \qquad (\alpha = a/d, \quad \beta = b/d), \tag{8.6}$$

and is sometimes called the *entire linear transformation*. The transformation (8.6) is defined for all values of z, and if $\alpha \neq 0$, its derivative $L'(z)$ is a nonzero constant, so that (8.6) is a conformal mapping of the whole z-plane. Under this transformation, the tangents to all curves in the z-plane are rotated through the same angle, equal to $\operatorname{Arg} \alpha$, and the magnification at every point equals $|\alpha|$. If $\alpha = 1$, then

$$\operatorname{Arg} \alpha = 2k\pi, \qquad |\alpha| = 1,$$

where k is an integer, and then both the rotation and expansion produce no effect. In this case, the transformation takes the form

$$w = z + \beta,$$

which obviously corresponds to displacing the whole plane by the vector β. On the other hand, if $\alpha \neq 1$ (and $\alpha \neq 0$), the transformation (8.6) can be written in the form

$$w - z_0 = \alpha(z - z_0),$$

where z_0 is determined from the equation[5]

$$z_0 = \alpha z_0 + \beta.$$

Then it is immediately clear that the transformation (8.6) is equivalent to a rotation of the whole plane through the angle $\operatorname{Arg} \alpha$ about the point $z_0 = \beta/(1 - \alpha)$, together with a uniform magnification by the factor $|\alpha|$ relative to the point z_0. This magnification is sometimes called a *homothetic transformation* (or *transformation of similitude*) with *ray center* z_0 and *ray ratio* $|\alpha|$.

Next suppose that $c \neq 0$ in (8.4). Then the derivative

$$L'(z) = \frac{ad - bc}{(cz + d)^2} = \frac{ad - bc}{c^2} \frac{1}{(z - \delta)^2}$$

exists, if $z \neq \delta$, where $\delta = -d/c$. If the determinant

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

vanishes, then its rows are proportional,[6] i.e.,

$$\frac{a}{c} = \frac{b}{d} = \mu \quad \text{or} \quad a = \mu c, \quad b = \mu d,$$

where μ is a constant, so that (8.5) reduces to the trivial transformation

$$L(z) = \frac{az + b}{cz + d} = \frac{\mu cz + \mu d}{cz + d} \equiv \mu.$$

If $ad - bc \neq 0$, then $L'(z) \neq 0$ for all $z \neq \delta$, and hence the mapping $w = L(z)$ is conformal at all finite points except possibly at $z = \delta$. Under the mapping, the tangents to curves passing through any point $z \neq \delta$ are rotated through an angle equal to

$$\operatorname{Arg} L'(z) = \operatorname{Arg} \frac{ad - bc}{c^2} - 2 \operatorname{Arg} (z - \delta),$$

while the magnification at z equals

$$|L'(z)| = \left|\frac{ad - bc}{c^2}\right| \frac{1}{|z - \delta|^2}.$$

The angle through which tangents are rotated has the same value for all points with equal values of $\operatorname{Arg} (z - \delta)$, i.e., along any ray drawn from δ, but otherwise varies from point to point. Similarly, in general the magnification varies with z, but it has the same value for all points with equal values of $|z - \delta|$, i.e., along any circle with center δ. In particular, the magnification is equal to 1 at every point of the circle C with equation

$$|z - \delta| = \frac{1}{|c|} \sqrt{|ad - bc|}$$

[5] Obviously, z_0 is invariant under the transformation (8.6), i.e., z_0 is a *fixed point* of the transformation. If $\alpha \neq 0$, the point at infinity is also a fixed point (see Sec. 46).

[6] See e.g., G. E. Shilov, *op. cit.*, p. 25.

(called the *isometric circle* of the Möbius transformation), is greater than 1 inside C (approaching ∞ as $z \to \delta$), and is less than 1 outside C (approaching 0 as $z \to \infty$). The situation is shown schematically in Figure 8.1.

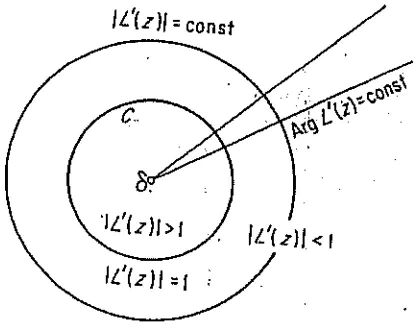

FIGURE 8.1

34. Conformal Mapping of the Extended Plane

As in the preceding section, let $c \neq 0$ and $ad - bc \neq 0$. Then it is clear that

$$\lim_{z\to\delta} \frac{az+b}{cz+d} = \infty, \qquad \lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c} = A,$$

and hence

$$L(\delta) = \infty, \qquad L(\infty) = A.$$

Thus the function $w = L(z)$ maps the finite point δ into the point at infinity (∞), and maps the point at infinity into the finite point A. We now show that the mapping is conformal at these points too. First let γ_1 and γ_2 be two curves forming an angle θ with its vertex at the point δ, and let Γ_1 and Γ_2 be their images in the w-plane. To prove that Γ_1 and Γ_2 form an angle θ with its vertex at infinity, we subject the w-plane to the transformation

$$\eta = \frac{1}{w}.$$

Then the curves Γ_1 and Γ_2 go into two curves Γ_1^* and Γ_2^*, and the point at infinity goes into the origin of coordinates (see Figure 8.2). Obviously, we can go from γ_1 and γ_2 in the z-plane to Γ_1^* and Γ_2^* in the η-plane by making the Möbius transformation

$$\eta = \frac{1}{w} = \frac{cz+d}{az+b},$$

which is conformal at the point $z = \delta = -d/c$. It follows that the curves Γ_1^* and Γ_2^* form an angle θ with its vertex at the origin. Therefore, according

to the definition given in Sec. 25, the curves Γ_1 and Γ_2 also form an angle θ with its vertex at infinity. This proves that the mapping $w = L(z)$ is conformal at the point $z = \delta$.

The fact that $w = L(z)$ is conformal at ∞ is proved similarly. In fact, if the curves γ_1 and γ_2 go through the point at infinity in the z-plane, their images Γ_1 and Γ_2 in the w-plane go through the point A. Suppose γ_1 and γ_2 form an angle θ with its vertex at infinity. This means that their images γ_1^*

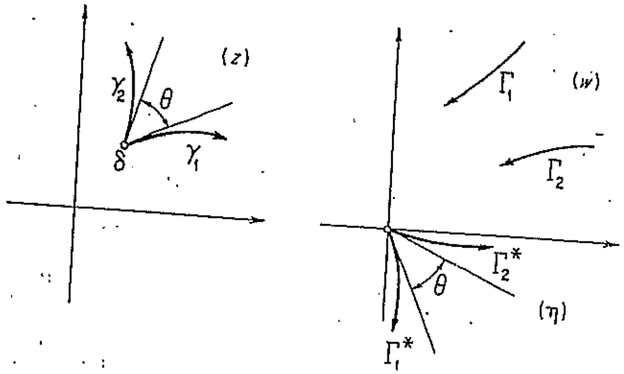

FIGURE 8.2

and γ_2^* under the transformation $\zeta = 1/z$ form an angle θ with its vertex at the origin. But we can obviously go from γ_1^* and γ_2^* to Γ_1 and Γ_2 by making the Möbius transformation

$$w = \frac{az+b}{cz+d} = \frac{a\dfrac{1}{\zeta}+b}{c\dfrac{1}{\zeta}+d} = \frac{a+b\zeta}{c+d\zeta},$$

which is conformal at the point $\zeta = 0$. It follows that Γ_1 and Γ_2 form the same angle θ at the point $A = a/c$. This proves that the mapping $w = L(z)$ is conformal at ∞. The situation can be summarized by saying that the *transformation $w = L(z)$ is a conformal mapping of the extended plane onto itself.*

Remark 1. These considerations suggest the following definition: A function $f(z)$ is said to be *analytic at $z = \infty$* if the function $f^*(\zeta) = f(1/\zeta)$ is analytic at $\zeta = 0$. In particular, if $f(z)$ is analytic at $z = \infty$, the limit

$$\lim_{z\to\infty} f(z) = \lim_{\zeta\to 0} f^*(\zeta) = f(\infty)$$

always exists and is finite. We define the derivative of $f(z)$ at $z = \infty$ to be the quantity

$$f'(\infty) = f^{*\prime}(0)$$

where it should be noted that in general

$$f'(\infty) \neq \lim_{z \to \infty} f'(z)$$

(see Problem 8.10). Then, by the argument given above for the special case of the Möbius transformation, it is easy to see that the mapping $w = f(z)$ is conformal at ∞ if $f'(\infty) \neq 0$. With this approach, the conformality at ∞ of the Möbius transformation

$$L(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0, \quad c \neq 0)$$

follows from the fact that

$$L'(\infty) = -\frac{ad - bc}{c^2} \neq 0.$$

Remark 2. Similarly, if

$$\lim_{z \to a} f(z) = \infty,$$

but if

$$\varphi(z) = \frac{1}{f(z)}$$

is analytic at $z = a$, with derivative $\varphi'(a) \neq 0$, then, just as in the case of the Möbius transformation, the mapping $w = f(z)$ is conformal at $z = a$.

PROBLEMS

8.1. With the same notation as on p. 118, a curve l is said to have a *left-hand tangent* (of inclination θ) at the point $z_0 = \lambda(t_0)$ if the limit (8.2) exists, subject to the extra condition that every point of the sequence $\{t_n\}$ converging to t_0 be less than t_0. The *right-hand tangent* is defined similarly by requiring that $t_n > t_0$ for every n. Give an example of a (continuous) curve l which has a left-hand tangent but no right-hand tangent (and hence no tangent) at a point $z_0 \in l$.

8.2. Verify that the function

$$f_1(z) = r^2(\cos \Phi + i \sin \Phi) = rz$$

used in Example 2, p. 121 is differentiable at $z = 0$.

8.3. Find the angle through which a curve drawn from the point z_0 is rotated under the mapping $w = z^2$ if

a) $z_0 = i$; b) $z_0 = -\frac{1}{4}$; c) $z_0 = 1 + i$; d) $z_0 = -3 + 4i$.

Also find the corresponding values of the magnification.

8.4. Carry out the same calculations as in the preceding problem, this time applied to the function $w = z^3$.

8.5. Which part of the plane is shrunk and which part stretched under the following mappings:

a) $w = z^2$; b) $w = z^2 + 2z$; c) $w = \dfrac{1}{z}$?

8.6. As shown on pp. 122–123, the entire linear transformation $w = \alpha z + \beta$ is equivalent to a rotation and a magnification relative to the fixed point $z_0 = \beta/(1 - \alpha)$, provided that $\alpha \neq 0$. Find the rotation, magnification and (finite) fixed point, if such exists, corresponding to each of the following transformations, and write each in the canonical form $w - z_0 = \alpha(z - z_0)$:

a) $w = 2z + 1 - 3i$; b) $w = iz + 4$;
c) $w = z + 1 - 2i$; d) $w - w_1 = a(z - z_1)$ $(a \neq 0)$.

8.7. Find the entire linear transformation with fixed point $1 + 2i$ carrying the point i into the point $-i$.

Ans. $w = (2 + i)z + 1 - 3i$.

8.8. Find the entire linear transformation carrying the triangle with vertices at the points $0, 1, i$ into the similar triangle with vertices at the points $0, 2, 1 + i$.

8.9. Prove that the transformation $L(z) = \alpha z + \beta$ is conformal at infinity if $\alpha \neq 0$.

8.10. Prove that if $f(z)$ is analytic at infinity, then

$$\lim_{z \to \infty} f'(z) = 0.$$