To summarize, there is an asymptotically stable equilibrium at $p^* = \frac{1}{2}$ unless the genotypes AA have a selective advantage of a factor $e^4 \approx 55$ over the heterozygotes Aa, and of $e^8 \approx 3000$ over the homozygotes aa (i.e., $\beta > 4$). In the latter case, when the frequency-dependent selective forces are so strong ($\beta > 4$), we obtain an (asymptotically) stable 2-cycle.

REMARK 1.3 In [34], it was shown that for $0 < \beta \neq 4$, the equilibrium point $p_3^* = \frac{1}{2}$ is in fact globally asymptotically stable on the interval (0, 1). ∎

Exercises - (1.9)

1. Show that $SF(\frac{1}{2}) < 0$, where F is the map defined by

$$F(p) = \frac{pe^{\beta(1-2p)}}{pe^{\beta(1-2p)} + 1 - p}.$$

2. Let $G(x) = x\exp[\beta\frac{(1-x)}{1+x}]$, $\beta > 4$, $x \in (0, \infty)$. Let $\{\overline{x}_1, \overline{x}_2\}$ be a 2-cycle of G. Show that this 2-cycle is asymptotically stable.

[5]

Chapter 2

Sharkovsky's Theorem and Bifurcation

Period three implies chaos.

Li and Yorke

2.1 The Mystery of Period 3

In 1975, Li and Yorke [39] published the article, "Period three implies chaos" in the *American Mathematical Monthly*. In this paper, they proved that if a continuous map f has a point of period 3, then it must have points of any period k. Soon afterward, it was found that Li–Yorke's theorem is only a special case of a remarkable theorem published in 1964 by the Ukranian mathematician Alexander Nikolaevich Sharkovsky [61]. Sharkovsky introduced a new ordering $\rhd$ of the positive integers in which 3 appears first. He proved that if $k \rhd r$ and f has a k-periodic point, then it must have an r-periodic point. This clearly implies Li–Yorke's theorem. However, to their credit, Li and Yorke were the first to coin the word "chaos" and introduce it to mathematics.

It is worth mentioning that neither Li–Yorke's theorem nor Sharkovsky's theorem is intuitive. To illustrate this point, recall from Example 1.10 that the tent map $T(x) = 1 - 2|x - \frac{1}{2}|$ has two cycles of period 3 : $\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$ and $\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\}$.

Is it intuitively clear that the tent map has cycles of all periods? I do not think so.

Let us now turn our attention to Sharkovsky's ordering of the positive integers. This ordering is defined as follows:

$3 \rhd 5 \rhd 7 \rhd \ldots$ odd integers	$2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \rhd \ldots$ $2\times$ odd integers
$2^2 \times 3 \rhd 2^2 \times 5 \rhd 2^2 \times 7 \rhd \ldots$ $2^2\times$ odd integers	$2^n \times 3 \rhd 2^n \times 5 \rhd 2^n \times 7 \rhd \ldots$ $2^n\times$ odd integers
$\ldots \ldots \ldots$	$\ldots \rhd 2^n \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$ powers of 2.

We first list all the odd integers except 1, then 2 times the odd integers, 2^2 times the odd integers, and, in general, 2^n times the odd integers for all $n \in \mathbb{Z}^+$. This is followed by powers of 2 in a descending order. It is easy to see that this ordering exhausts all of the positive integers. Notice that $m \rhd n$ signifies that m appears before n in the Sharkovsky's ordering.

THEOREM 2.1
(Sharkovsky's Theorem). *Let $f : I \to I$ be a continuous map on the interval I, where I may be finite, infinite, or the whole real line.*

If f has a periodic point of period k, then it has a periodic point of period r for all r with $k \rhd r$.

PROOF See the Appendix at the end of this chapter. Proof of the theorem may also be found in Block and Coppel [7].

We will now make a few comments about the theorem and then give a proof of a consequence of it: the Li–Yorke theorem.

1. The only way that a continuous map f has finitely many periodic points is if f has only periods that are powers of 2. Otherwise, it has infinitely many periodic points. For example, if f has a periodic point of period $2^{10} \times 5$, then it has infinitely many periodic points of periods

$$2^{10} \times 5, 2^{10} \times 7, 2^{10} \times 9, \ldots 2^{11} \times 3, 2^{11} \times 5, 2^{11} \times 7, \ldots$$
$$\ldots 2^n, 2^{n-1}, \ldots, 2^2, 2, 1.$$

2. If $m \rhd n$, then there are continuous maps with periodic points of period n but not of period m (see the proof of Theorem 2.3).

3. Sharkovsky's theorem does not extend to two or higher dimensional Euclidean spaces. It is not even true for the unit circle S^1. For example, the map $f : S^1 \to S^1$ defined by $f(e^{i\theta}) = e^{i(\theta + \frac{2\pi}{3})}$ is of period 3 at all points in S^1, but f has no other periods. ∎

Now we go back and prove the Li–Yorke theorem.

THEOREM 2.2
(Li and Yorke). *Let $f : I \to I$ be a continuous map on an interval I. If there is a periodic point in I of period 3, then for every $k = 1, 2, \ldots$ there is a periodic point in I having period k.*

To prove this theorem, we need some preliminary results.

LEMMA 2.1
Let $f : I \to R$ be continuous, where I is an interval. For any closed interval $J \subset f(I)$, there is a closed interval $Q \subset I$ such that $f(Q) = J$.

PROOF Let $J = [f(p), f(q)]$, where $p, q \in I$. If $p < q$, let r be the largest number in $[p, q]$ with $f(r) = f(p)$ and let s be the smallest number in $[p, q]$ such that $f(s) = f(q)$ and $s > r$. We claim that $f([r, s]) = J$. We observe that by the intermediate value theorem,[1] we have $f([r, s]) \supset J$. Assume that there exists t with $r < t < s$ such that $f(t) \notin J$. Without loss of generality, suppose that $f(t) > f(q)$. Applying the intermediate value theorem again yields $f([r, t]) \supset J$. Hence, there is $x \in [r, t)$ such that $f(x) = f(q)$, which contradicts our assumption that s is the smallest number in $[p, q]$ with $f(p) = f(q)$. The case where $p > q$ is similar. The proof is now complete. ∎

LEMMA 2.2
Let $f : I \to I$ be continuous and let $\{I_n\}_{n=0}^{\infty}$ be a sequence of closed and bounded intervals with $I_n \subset I$ and $I_{n+1} \subset f(I_n)$ for all $n \in \mathbb{Z}^+$. Then, there is a sequence of closed and bounded intervals Q_n such that $Q_{n+1} \subset Q_n \subset I_0$ and $f^n(Q_n) = I_n$ for $n \in \mathbb{Z}^+$.

PROOF Define $Q_0 = I_0$. Then, $f^0(Q_0) = I_0$. If Q_{n-1} has been defined so that $f^{n-1}(Q_{n-1}) = I_{n-1}$, then $I_n \subset f(I_{n-1}) = f^n(Q_{n-1})$. By applying Lemma 2.1 on f^n, there is a closed bounded interval $Q_n \subset Q_{n-1}$ such that $f^n(Q_n) = I_n$. ∎

[1] If f is continuous on [a, b] and N is any number between $f(a)$ and $f(b)$, then there is at least one c between a and b such that $f(c) = N$.

We are now well prepared to give the proof of Theorem 2.2.

Proof of Theorem 2.2 Suppose that f has a 3-cycle $\{x, f(x), f^2(x)\}$. Then one may rename the elements of the cycle so that it will become $\{a, b = f(a), c = f(b)\}$ with either $a < b < c$ or $a > b > c$. For example, if $x < f^2(x) < f(x)$, we let $a = f(x), b = f(a), c = f^2(a)$ and thus we have $a > b > c$. Let us assume that $a < b < c$. Write $J = [a, b], L = [b, c]$. For any positive integer $k > 1$, let $\{I_n\}$ be a sequence of intervals with $I_n = L$ for $n = 0, 1, \ldots, k-2$ and $I_{k-1} = J$, and define I_n to be periodic inductively, $I_{n+k} = I_n$ for $n \in \mathbb{Z}^+$. The sequence $\{I_n\}$ looks like

$$\underbrace{L, L, \ldots, L}_{(k-1) \text{ times}}, J, \underbrace{L, L, \ldots, L}_{(k-1) \text{ times}}, J, \underbrace{L, L, \ldots, L}_{(k-1) \text{ times.}}, J, \ldots$$

If $k = 1$, let $I_n = L$ for all $n \in \mathbb{Z}^+$. Since $f(a) = b, f(b) = c$, and $f(c) = a$, it follows by the intermediate value theorem that $L, J \subset f(L)$ and $L \subset f(J)$ (see Fig. 2.1). ∎

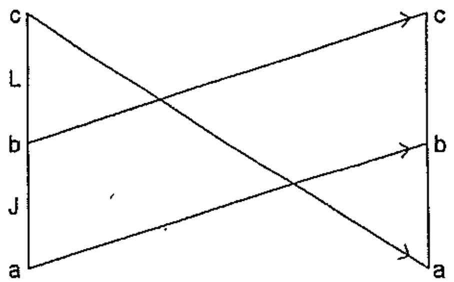

FIGURE 2.1
$f(a) = b, f(b) = c, f(c) = a$. **Hence** $J \subset f(L)$ **and** $L \subset f(J)$.

Hence, one may apply Lemma 2.2 to produce a sequence $\{Q_n\}$ of closed, bounded intervals with $Q_k \subset Q_0 = L$ and $f^k(Q_k) = I_k = L$. Consequently, $L \subset f^k(L)$. By applying Theorem 1.1 to f^k, we conclude that f^k has a fixed point in L and, consequently, f has a k-periodic point in I.

2.2 Converse of Sharkovsky's Theorem

The question that we are going to address in this section is the following: given any positive integers k and r with $k \rhd r$, is there a continuous map that has a point of period r but no points of period k? The answer to this question is a definite yes. Here we give a simple proof of this result which is based on our paper [21].

THEOREM 2.3
(A Converse of Sharkovsky's Theorem). *For any positive integer r, there exists a continuous map $f_r : I_r \to I_r$ on the closed interval I_r such that f_r has a point of prime period r but no points of prime periods s, for all positive integers s that precede r in the Sharkovsky's ordering, i.e., $s \rhd \ldots \rhd r$.*

PROOF In order to accomplish the proof, we have three cases to contemplate.

1. Odd periods
2. Periods of the form $2^n \times$ odd positive integers
3. Periods of powers of 2, i.e., 2^n

Case 1: Odd Periods.

(a) Let us construct a continuous map that has points of period 5 but no points of period 3. Define a map $f : [1, 5] \to [1, 5]$ as follows:

$$f(1) = 3,\ f(2) = 5,\ f(3) = 4,\ f(4) = 2, \text{ and } f(5) = 1\,.$$

On each interval $[n, n+1]$, $1 \le n \le 4$, we assume f to be linear (see Fig. 2.2).

Observe first that none of the points $1, 2, 3, 4, 5$ is a 3-periodic point; they all belong to the single 5-cycle: $1 \xrightarrow{f} 3 \xrightarrow{f} 4 \xrightarrow{f} 2 \xrightarrow{f} 5 \xrightarrow{f} 1$. Note also that

$$f^3([1, 2]) = [2, 5],\ f^3([2, 3]) = [3, 5], \text{ and } f^3([4, 5]) = [1, 4]\,.$$

Hence, f^3 has no fixed points in the intervals $[1, 2]$, $[2, 3]$, and $[4, 5]$. The situation with the interval $[3, 4]$ is much more involved since $f^3([3, 4]) =$

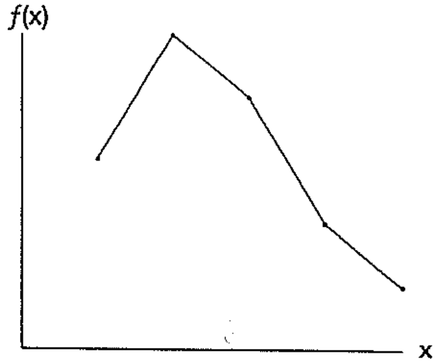

FIGURE 2.2
A map of period 5 but no points of period 3.

$[1, 5]$. This implies by Theorem 1.1 that f^3 must have a fixed point $\bar{x}$ in the interval $[3, 4]$. We must show now that this fixed point of f^3 is really a fixed point of f and thus is not of prime period 3. Observe that $f(\bar{x}) \in [2, 4]$. So, if $f(\bar{x}) \in [2, 3]$, then $f^2(\bar{x}) \in [4, 5]$ and $f^3(\bar{x}) \in [1, 2]$. But, this is impossible since $f^3(\bar{x}) = \bar{x} \in [3, 4]$. Therefore, we conclude that $f(\bar{x}) \in [3, 4]$. Note that $f^2(\bar{x}) \in [2, 4]$. Again, if $f^2(\bar{x}) \in [2, 3]$, then $f^3(\bar{x}) \in [4, 5]$, yet another contradiction. Thus, the orbit of $\bar{x}$, $\{\bar{x}, f(\bar{x}), f^2(\bar{x})\}$ is a subset of the interval $[3, 4]$.

Now, on the interval [3,4] $f(x) = 10 - 2x$ has the unique fixed point $x^* = \frac{10}{3}$. Moreover, on [3,4] $f^3(x) = 30 - 8x$ also has the unique fixed point $\bar{x} = \frac{10}{3} = x^*$. Hence, f has no points of prime period 3.

(b) One may generalize the above construction in order to manufacture continuous maps that have points of period $2n + 1$ but no points of period $2n - 1$. Details will be given in the problems (Problems 3, 4, and 5).

Case 2: Periods of the Form $2^k(2n + 1)$.

(a) We begin by constructing a map that has points of period 2×5 but has no points of period 2×3. Consider first the map $f : [1, 5] \to [1, 5]$ as defined in Case 1(a). This map has points of period 5 but has no points of period 3. We will use this map to construct a new map $\tilde{f}$, called **the double** of f, as follows:

$$\tilde{f} : [1, 13] \to [1, 13],$$

$$\tilde{f}(x) = \begin{cases} f(x) + 8; & 1 \le x \le 5 \\ x - 8; & 9 \le x \le 13. \end{cases}$$

For $5 < x < 9$, we connect the points (5, 9) and (9, 1) by a line (Fig. 2.3). The proof that the double map $\tilde{f}$ has a 10-cycle but no 6-cycle is left to the reader as Problem 6.

FIGURE 2.3
A 10-cycle but no 6-cycles.

(b) The general procedure for constructing the double $\tilde{f}$ of any map $f : [1, 1 + h] \to [1, 1 + h]$ is as follows: $\tilde{f} : [1, 1 + 3h] \to [1, 1 + 3h]$, where

$$\tilde{f}(x) = \begin{cases} f(x) + 2h; & 1 \le x \le 1 + h \\ x - 2h; & 1 + 2h \le x \le 1 + 3h \end{cases}$$

and $\tilde{f}$ is linear for $1 + h < x < 1 + 2h$. So, if we want to construct a map with points of period $2(2n + 1)$ but no points of period $2(2n - 1)$, $n = 3, 4, 5, \ldots$, we start with a map f that has points of period $(2n + 1)$ but no points of period $(2n - 1)$. Then, its double map $\tilde{f}$ will have the desired properties (Problem 7).

Case 3: Periods of the Form 2^n.

(a) It is easy to construct a map that has points of period $2^0 = 1$ (fixed points) but no points of prime period 2^1. Just pick any map $f(x) = ax + b$ with $a \neq \pm 1$. To construct a map that has points of period 2 but no points of period 2^2, we consider the map $f(x) = -x + b$. Then, $x = \frac{b}{2}$ is a fixed point of f. However, $f^2(x) = -(-x + b) + b = x$. Thus, every point, with the exception of $x^* = \frac{b}{2}$, is of prime period 2.

(b) To construct a map that has points of period 2^2 but no points of period 2^3, we use the double map $\tilde{f}$ of the map $f(x) = -x + 3$ (see Fig. 2.4). Map doubling may be used repeatedly to construct maps with 2^n-cycles but no 2^{n+1}-cycles. ▮

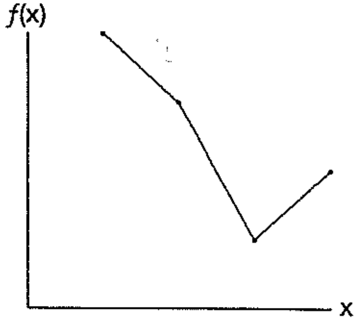

FIGURE 2.4
A 4-cycle but no 8-cycles.

Unresolved questions that remain to be settled are as follows:

1. Can we construct a continuous map that has points of period $2^n \times 3$ but has no points of any period of the form $2^{n-1} \times$ odd integer (see Problem 12)?
2. Can we construct a continuous map that has points of period 2^n for all $n \in \mathbb{Z}^+$ but no points of any other period [2] (see Problems 12 and 13)?

Exercises - (2.1 and 2.2)

1. Show that the piecewise linear map $g : [1, 7] \to [1, 7]$ shown in Fig. 2.5 has a 7-cycle but does not have a 5-cycle.
2. Mimic Problem 1 to construct a map that has a 9-cycle but not a 7-cycle.
3. Construct a map that has a $(2k + 1)$-cycle but has no $(2k - 1)$-cycle for any $k > 3$.
4. Consider the map f defined in Fig. 2.2 on the interval $I = [1, 5]$. Define a new function $\tilde{f}$ on $J = [1, 13]$ (called the **double** of f) by compressing

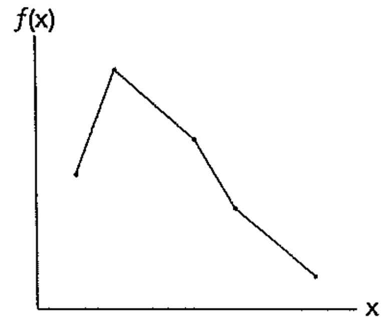

FIGURE 2.5
A 7-cycle but no 5-cycles.

the graph of f into the upper left square. Explicitly, we let

$$\tilde{f}(x) = \begin{cases} f(x) + 8 & \text{for } 1 \le x \le 5 \\ x - 8 & \text{for } 9 \le x \le 13\,. \end{cases}$$

Then we connect the points (5, 9) and (9, 1) by a line. Show that the map $\tilde{f}$ (Fig. 2.3) has a 10-cycle but not a 6-cycle.

5. Mimic Problem 4 to produce a map with a 14-cycle but not a 10-cycle.
6. Construct a map that has a $2(2n + 1)$-cycle but no $2(2n - 1)$-cycles.
7. Let f be a map defined on the interval $I = [1, 1 + h]$. Define $\tilde{f}$, "the double of f," on $[1, 1 + 3h]$ as follows:

$$\tilde{f}(x) = \begin{cases} f(x) + 2h & \text{for } 1 \le x \le 1 + h \\ x - 2h & \text{for } 1 + 2h \le x \le 1 + 3h \end{cases}$$

and filling the rest of the graph as in Fig. 2.5. Prove that $\tilde{f}$ has a $2n$-periodic point at x if and only if f has an n-periodic point at x. Show that if f has points of period $2^k(2n + 1)$, then $\tilde{f}$ has points of period $2^{k+1}(2n + 1)$.
8. Construct a map that has an 8-cycle but no 16-cycle.
9. Construct a map that has a 2^k-cycle but no 2^{k+1}-cycle, for $k > 3$.
10. (a) Construct a continuous map that has a point of period 2×3 but no points of odd periods.

(b) Describe the procedure of constructing a map of period $2^n \times 3$ but has no points of period $2^{n-1} \times$ odd integer.

For another construction of the double map on the same interval: Let $I = [0, 1]$ and $f : I \to I$ be continuous. Define the double map $\tilde{f}$ by

$$\tilde{f}(x) = \begin{cases} \frac{2}{3} + \frac{1}{3} f(3x) & \text{for } 0 \le x \le \frac{1}{3} \\ [2 + f(1)](\frac{2}{3} - x) & \text{for } \frac{1}{3} \le x \le \frac{2}{3} \\ x - \frac{2}{3} & \text{for } \frac{2}{3} \le x \le 1 . \end{cases}$$

11. Show that $\tilde{f}$ has a $2n$-periodic point at x if and only if f has an n- periodic point at x.

12*. Use Problem 11 to construct a continuous map that has fixed points of period 2^n for all $n \in \mathbb{Z}^+$, but has no points of any other period.

(Hint: Start with $f(x) = \frac{1}{3}$ on $[0, 1]$. Let $f_1 = \tilde{f}$ by its double map, $f_2 = \tilde{f}_1, \ldots, f_n = \tilde{f}_{n-1}$. Define $f_\infty(x) = \lim_{n\to\infty} f_n(x)$. Show that f_∞ is continuous and has points of period 2^n for all n and no other periods.)

13*. Construct a continuous map that has points of period 2^n for all $n \in \mathbb{Z}^+$ but has no points of any other period.

14*. Generalize the Li–Yorke theorem (Theorem 2.2) as follows: Let J be an interval and let $f : J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = f(a)$, $c = f^2(a)$, and $d = f^3(a)$, satisfy $d \le a < b < c$ ($d \ge a > b > c$). Prove that for every $k = 1, 2, \ldots$ there is a periodic point in J having period k.

15. Let f be a continuous map on the interval $[a, b]$. If there exists a point $x_0 \in [a, b]$ with $f^2(x_0) < x_0 < f(x_0)$, or $f(x_0) < x_0 < f^2(x_0)$, prove that f has a 2-cycle in $[a, b]$.

16. Prove that a homeomorphism of R cannot have periodic points with prime period greater than 2. Give an example of a homeomorphism that has a point of prime period 2.

17*. (Li and Yorke) [39]. Under the assumption of Problem 14, show that there is an uncountable set $S \subset J$, containing no periodic points, which satisfies the following conditions:

(a) For every $x, y \in S$ with $x \neq y$,

$$\lim_{n\to\infty} \sup |F^n(x) - F^n(y)| > 0$$

and

$$\lim_{n\to\infty} \inf |F^n(x) - F^n(y)| = 0 .$$

(b) For every $y \in S$ and periodic point $q \in J$,

$$\lim_{n\to\infty} \sup |F^n(x) - f^n(q)| > 0 .$$

2.3 Basin of Attraction

It is customary to call an asymptotically stable fixed point or a cycle an **attractor**. This name makes sense because the orbits of all nearby points tend to the attractor. The maximal set that is attracted to an attractor M is called the **basin of attraction** of M. Our analysis here applies to cycles of any period, but for simplicity we will restrict our attention to attracting fixed points.

DEFINITION 2.1 *Let x^* be an asymptotically stable fixed point of a map f. Then the* **basin of attraction** *(or the stable set) $W^s(x^*)$ of x^* is defined as the maximal interval J containing x^* such that if $x \in J$, then $f^n(x) \to x^*$ as $n \to \infty$.*

Observe that from the definition of an attractor, $W^s(x^*)$ contains an open interval around x^*.

Example 2.1

1. The map $f(x) = x^2$ has one attracting fixed point $x^* = 0$. Its basin of attraction $W^s(0) = (-1, 1)$. Note that 1 is a fixed point and -1 is an eventually fixed point that goes to 1 after the first iteration.

2. The logistic map $F_{2.5}(x) = 2.5x(1 - x)$ has one attracting fixed point $x^* = 0.6$ whose basin of attraction is $W^s(0.6) = (0, 1)$. □

It is worth noting here that finding a basin of attraction of a fixed point is in general a difficult task. The most efficient method to determine the basin