Chapter 7

The Julia and Mandelbrot Sets

A manifesto: There is a fractal face to the geometry of nature.

Benoit Mandelbrot

7.1 Mapping by Functions on the Complex Domain

Let $z = x + iy$ be a complex number and let $\mathbb{C}$ denote the set of complex numbers. Then x is called the real part of z, $\Re(z)$, and y is called the imaginary part of z, $\Im(z)$. Note that both x and y are real numbers. If we let the x axis to be the real axis and the y axis to be the imaginary axis, then the complex number $z = x + iy$ is represented by the point (x, y) in this complex plane (see Fig. 7.1). The modulus $|z|$ of z is defined as $|z| = \sqrt{x^2 + y^2}$; it is the distance between z and the origin. A complex number $z = x + iy$ may be represented in polar coordinates. Let $r = |z|$, and $\theta = \tan^{-1}(\frac{y}{x})$. Then θ is called the argument of z, denoted by $\arg(z)$. Moreover, $z = re^{i\theta}$. It is noteworthy to observe that $|z| = r|e^{i\theta}| = r$, since $|e^{i\theta}| = |\cos\theta + i\sin\theta| = 1$.

The triangle inequality that we encountered in the real number system still holds for complex numbers.

Triangle Inequality for Complex Numbers

Let $z_1, z_2 \in \mathbb{C}$. Then

(a) $|z_1 + z_2| \leq |z_1| + |z_2|$

(b) $|z_1 + z_2| \geq |z_1| - |z_2|$

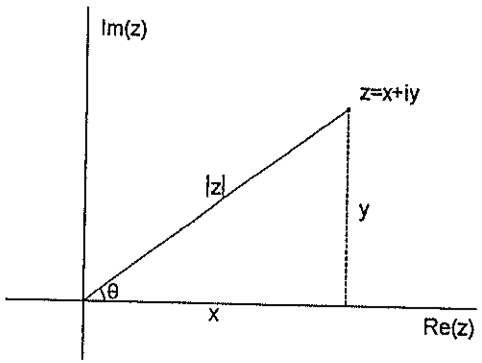

FIGURE 7.1
The modulus of a complex number $|z|$ and its argument θ.

Consider a linear map $f : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, of the form $f(z) = \alpha z$, where $\alpha = a + ib$, and $z = x + iy$.
Now, α and z may be written in the following exponential forms:

$$\alpha = se^{i\beta}, \text{ with } s = \sqrt{a^2 + b^2}, \text{ and } \beta = \tan^{-1}\left(\frac{b}{a}\right).$$

$$z = re^{i\theta}, \text{ with } r = \sqrt{x^2 + y^2}, \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

We may write $f(z)$ as $f(z) = sre^{i(\theta+\beta)}$.
Note also that $f^2(z) = s^2 re^{i(\theta+2\beta)}$, and generally

$$f^n(z) = s^n re^{i(\theta+n\beta)} . \qquad (7.1)$$

Clearly, we have three cases to consider:

1. $s < 1$: In this case, it follows from Eq. (7.1) that the orbit of z will spiral toward the origin. We may say, then, that the origin is asymptotically stable.

2. $s > 1$: From Eq. (7.1) we conclude that the orbit of z spirals further away from the origin and thus the origin is unstable.

3. $s = 1$: In this case the orbit of z stays on the circle of radius r_0 and the map is a rotation on the circle. Recall that we have discussed this map in Chapter 3. It was shown that if β is rational, then every point on a circle of radius r_0 is periodic, and if β is irrational, then the map on each circle of radius r_0 is transitive, with the set of periodic points dense but not chaotic.

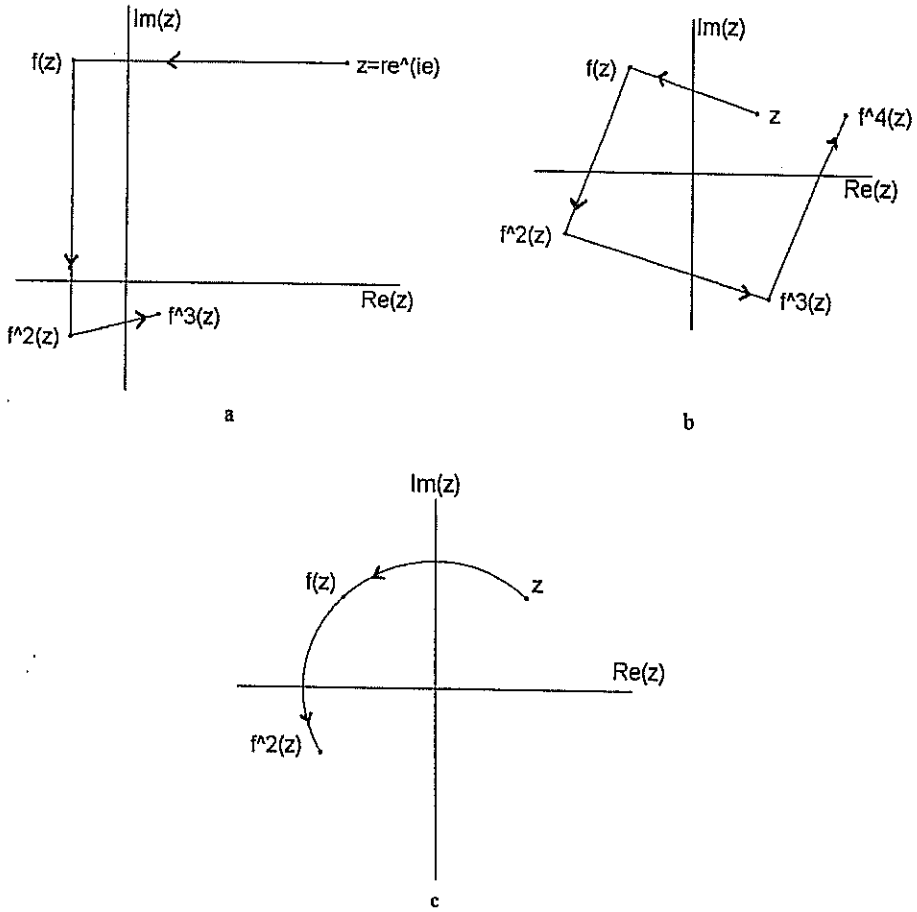

FIGURE 7.2
Iteration of a point $z = re^{i\theta}$ under the map $f(z) = \alpha z$, $\alpha = a + ib$, $s = \sqrt{a^2 + b^2}$. (a) $s < 1$: origin is asymptotically stable, (b) $s > 1$: origin is unstable, (c) $c = 1$: origin is stable.

Next, we consider more complicated nonlinear maps.

Example 7.1
Consider the squaring map $Q_0(z) = z^2$. Then for $z = re^{i\theta}$, $Q_0(z) = r^2 e^{i2\theta}$. Note that this function maps the upper half plane $r \geq 0, 0 \leq \theta \leq \pi$ onto the entire complex plane (Fig. 7.3). □

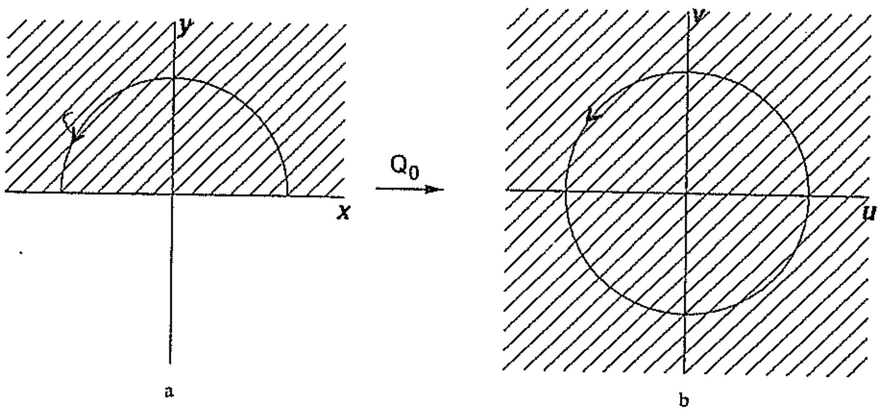

FIGURE 7.3
The map $Q_0(z) = z^2$ maps the upper half plane onto the entire complex plane.

Now, if we let $z = x + iy$ and $w = Q_0(z) = u + iv$, then $u + iv = x^2 - y^2 + i2xy$. Thus,

$$u = x^2 - y^2, \qquad v = 2xy\,. \tag{7.2}$$

Hence, each branch of the hyperbola $x^2 - y^2 = a$, $(a > 0)$ is mapped in a one-to-one manner onto the vertical line $u = a$. To see this, we note from the first part of Eq. (7.2) that $u = a$ if (x, y) is a point on one of the two branches of the hyperbola. When in particular it lies on the right-hand branch, the second part of Eq. (7.2) tells us that $v = 2y\sqrt{y^2 + a}$. Thus, the image of the right-hand branch can be expressed parametrically as

$$u = a, \qquad v = 2y\sqrt{y^2 + a}, \qquad -\infty < y < \infty$$

and is evident that the image of a point (x, y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction (Fig. 7.4).

Similarly,

$$u = a, \qquad v = -2y\sqrt{y^2 + a}$$

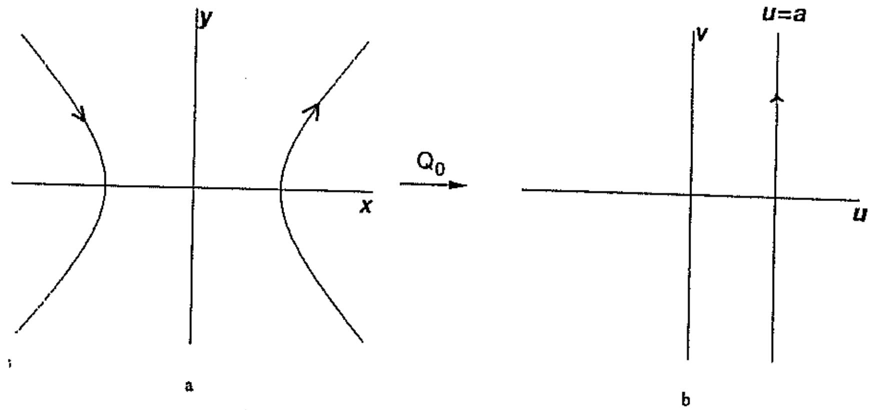

FIGURE 7.4
Q_0 maps a hyperbola to the line $u = a$.

furnishes a parametric representation for the image of the left-hand branch of the hyperbola. Thus, the left-hand branch of the hyperbola is mapped to the line $u = a$.

Let us now turn our attention to the analysis of the dynamics of the map Q_0. Clearly, $Q_0^n(z) = r^{2^n} e^{i\,2^n\theta}$. Furthermore, $|Q_0^n(z)| = r^{2^n}$. Consequently, we conclude that (see Fig. 7.5)

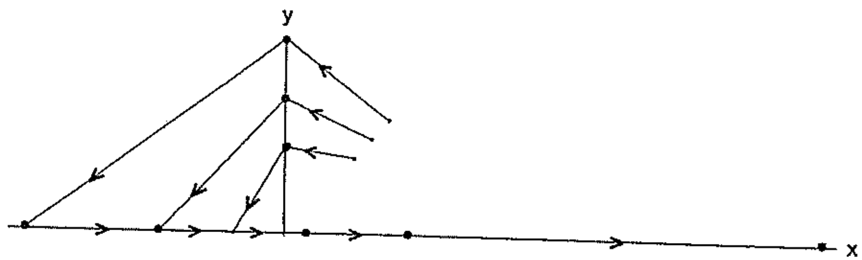

FIGURE 7.5
Orbits of the points $0.8e^{i\pi/4}$, $e^{i\pi/4}$, and $1.2e^{i\pi/4}$ under iteration of $Q_0(z) = z^2$.

1. $|Q_0^n(z)| \to 0$ as $n \to \infty$ if $r < 1$ or $(|z| < 1)$.
2. $|Q_0^n(z)| \to \infty$ as $n \to \infty$ if $r > 1$ or $(|z| > 1)$.
3. $|Q_0^n(z)| = 1$ if $r = 1$ or $(|z| = 1)$.

Next we consider the square root map.

Example 7.2
Consider the function $f(z) = z^{1/2}$.
If $z = r\,e^{i\theta}$ $(r > 0, \quad -\pi < \theta \leq \pi)$, then

$$z^{1/2} = \sqrt{r}\,e^{i\frac{(\theta+2k\pi)}{2}}, \qquad k = 0, 1\,. \quad \square \tag{7.3}$$

The principal branch of the double-valued function $z^{1/2}$ is given by $f_0(z) = \sqrt{r}\,e^{i\,\theta/2}$, $-\pi < \theta \leq \pi$, $r > 0$. Note that the origin and the ray $\theta = \pi$ form the branch cut for f_0, and the origin is the branch point.
From Eq. (7.3) the two square roots of z are

$$\begin{aligned} z_1 &= \sqrt{r}\,(\cos(\theta/2) + i\,\sin(\theta/2)) \\ z_2 &= \sqrt{r}\,\left(\cos\left(\frac{\theta}{2} + \pi\right) + i\,\sin\left(\frac{\theta}{2} + \pi\right)\right) \\ &= -\sqrt{r}\,(\cos(\theta/2) + i\,\sin(\theta/2)) \end{aligned}$$

(see Fig. 7.6).

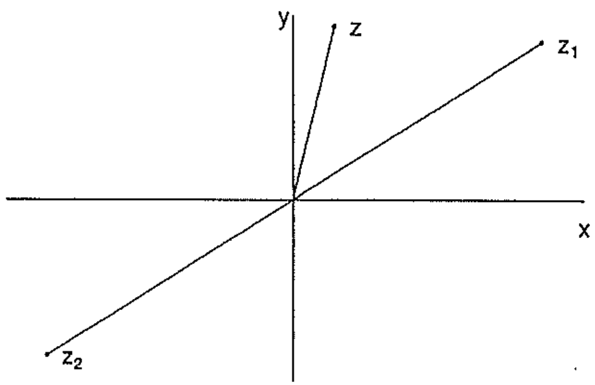

FIGURE 7.6
$z = re^{i\theta}$ **and its two square roots** z_1 **and** z_2**.**

If S^1 is a circle of radius r and center at the origin, then $f(S^1)$ is another circle of radius $\sqrt{r}$ centered at the origin [Fig. 7.7a(a)].
The situation is entirely different when the circle S^1 does not contain the origin. Here the circle lies in a wedge $\theta_1 \leq \theta \leq \theta_2$. Hence, the argument of each point in $f(S^1)$ lies in the wedge $\theta_1/2 \leq \theta \leq \theta_2/2$ and its reflection with

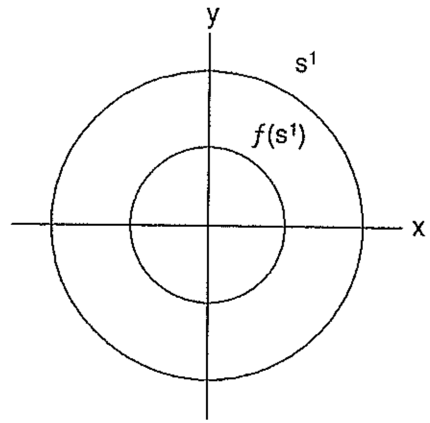

FIGURE 7.7a
(a) The image of S^1 **when it is centered at the origin.**

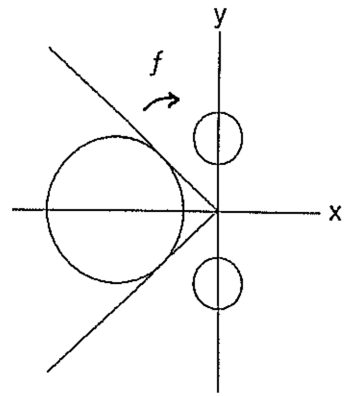

FIGURE 7.7b
(b) The image of S^1 **when it is not centered at the origin.**

respect to the origin. Hence, $f(S)$ is the union of two closed curves as shown in Fig. 7.7a(b). Observe that when the circle S^1 touches the origin, $f(S^1)$ looks like figure eight [Fig. 7.7c(c)]. Finally, when S^1 encircles the origin, $f(S^1)$ looks like a peanut shell [Fig. 7.7(d)].

A set D in the complex plane $\mathbb{C}$ is called a **domain** if it is open and connected.

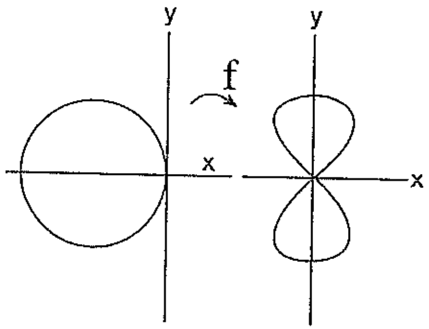

FIGURE 7.7c
(c) The image of S^1 when it passes through the origin.

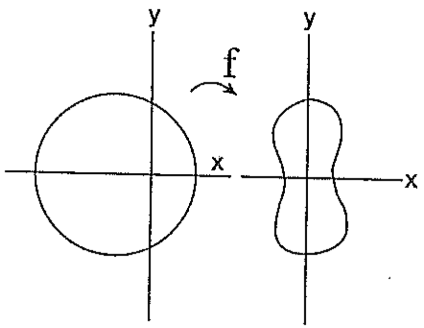

FIGURE 7.7d
(d) The image of S^1 when it encircles the origin.

DEFINITION 7.1 *A function f is said to be analytic in a domain D if it has a derivative at every point in D.*

We now state the main stability theorem for complex functions.

THEOREM 7.1
Let z^* be a fixed point of an analytic complex function f. Then, the following statements hold:

1. If $|f'(z^*)| < 1$, then z^* is asymptotically stable.
2. If $|f'(z^*)| > 1$, then z^* is unstable.

PROOF The proof is similar to that of Theorem 1.3 in Chapter 1 and will be left to the reader as Problem 10. ∎

As an immediate consequence of Theorem 7.1, we have the following result:

COROLLARY 7.1
Let z be a k-periodic point of an analytic function f. Then the following statements hold:

1. *If $|f'(z)f'(f(z))\dots f'(f^{k-1}(z))| < 1$, then z is asymptotically stable.*
2. *If $|f'(z)f'(f(z))\dots f'(f^{k-1}(z))| > 1$, then z is unstable.*

Example 7.3
Consider the map $f(z) = z^3,\ z \in C$.

(a) Find the fixed points of f and determine their stability.

(b) Find the 2-cycles of f and determine their stability. □

SOLUTION

(a) Fixed points: $z^3 - z = z(z^2 - 1) = 0$. Hence, the fixed points are: $z_1^* = 0,\ z_2^* = 1,\ z_3^* = -1$. Since $|f'(z_1^*)| = 0,\ |f'(z_2^*)| = 3,$ and $|f'(z_3)| = 3$, it follows from Theorem 7.1, that 0 is asymptotically stable, while 1 and -1 are unstable.

(b) To find the 2-cycles, we solve the equation $f^2(z) = z$. Hence, $z^9 - z = z(z^8 - 1) = 0$. Since 0 is a fixed point, we have $z^8 = 1$. Thus, the 2-cycles of $f(z)$ are the eighth roots of 1, excluding 1 and -1 since they are fixed points of f. The eighth roots of 1 are $w,\ w^2,\ w^3,\ w^5,\ w^6,\ w^7$, where $w = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. Note that $w^4 = -1$ is excluded from this list. Hence, the 2-cycles are $\{w, w^3\},\ \{w^2, w^6\},$ and $\{w^5, w^7\}$. It follows from Corollary 7.1 that all 2-cycles are unstable. ∎

7.2 The Riemann Sphere

To simplify the study of the dynamics of analytic maps, it is beneficial to consider the extended complex plane $\mathbb{C} \cup \{\infty\}$. To describe the topology of this space, we introduce a special representation of the complex plane. Consider the sphere S^2 with radius $\frac{1}{2}$ and center $(0, 0, \frac{1}{2})$ that is tangent to the complex plane $\mathbb{C}$ at the origin $(0, 0, 0)$. The point $N(0, 0, 1)$ will be referred to as the north pole of S^2 (see Fig. 7.8). We now introduce the stereographic projection S.

Let $P(a, b, c)\varepsilon S^2\backslash\{N\}$. The line joining N to P will pierce $\mathbb{C}$ at the point $Q(a/(1-c), b/(1-c), 0)$, which corresponds to the complex number $z = (a+ib)/(1-c)$ (Problem 6). Conversely, any point $Q(x, y, 0)$ in $\mathbb{C}$ corresponding to the complex number $z = x + iy$ lies on a line passing through the point N and intersecting the sphere S^2 at a point $P(\alpha, \beta, \gamma)$ with

$$\alpha = \frac{x}{x^2 + y^2 + 1}, \ \beta = \frac{y}{x^2 + y^2 + 1}, \ \gamma = \frac{x^2 + y^2}{x^2 + y^2 + 1}$$

(see Problem 11).

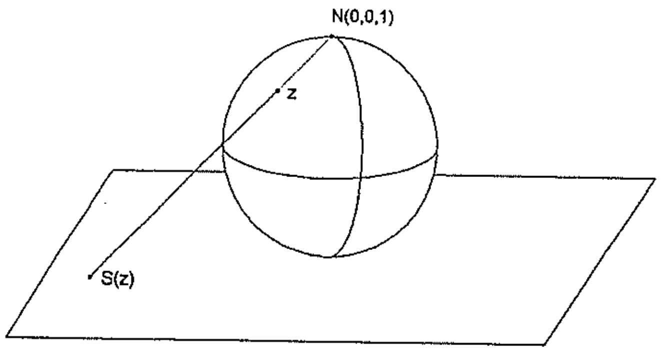

FIGURE 7.8
The Riemann Sphere: Stereographic projection from S^2 into $\mathbb{C}$.

Note that this gives a correspondence W from $\mathbb{C}$ onto $S^2\backslash\{N\}$. We then let $W(\infty) = N$. Hence, the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is identified with the sphere S^2, and either one will be called a Riemann sphere.

For any $z_0 \neq \infty$, we define an open ball $B_\varepsilon(z_0)$ as $B_\varepsilon(z_0) = \{z\varepsilon\mathbb{C} : |z - z_0| < \varepsilon\}$. We define open balls around ∞ as follows: For any $\varepsilon > 0$, we let $B_\varepsilon(\infty) = \{z\varepsilon\mathbb{C} : |z| > \frac{1}{\varepsilon}\}$. Note that the W takes $B_\varepsilon(\infty)$ to an open neighborhood of the north pole $N(0, 0, 1)$. The above description of open balls determines a metric on the extended complex place $\overline{\mathbb{C}}$.

Linear Fractional Transformation (Möbius Transformation)

The transformation $T(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, where a, b, c, d are complex constants, is called a linear fractional transformation (or Möbius transformation). The map T may be extended to $\overline{\mathbb{C}}$ by letting $T(\infty) = a/c$, and $T(z_0) = \infty$ when $cz_0 + d = 0$. An important property of the map T is that it maps circles in $\overline{\mathbb{C}}$ to circles. Note that a line in the complex plane $\mathbb{C}$ becomes a circle through ∞ in the extended complex place $\overline{\mathbb{C}}$. Hence, T maps lines and circles in $\mathbb{C}$ to lines and circles in $\overline{\mathbb{C}}$.

Exercises - (7.1 and 7.2)

1. If $z = x + iy$, then we may write $z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$. The two square roots of z are given by

$$\pm\sqrt{r}(\cos(\theta/2) + i\sin(\theta/2)).$$

Find the square roots and then draw them on the complex plane.

(a) i
(b) $-1 + i$
(c) $-1 + \sqrt{3}i$
(d) $1 + i$
(e) $-2 + 2i$
(f) -6

2. Plot the orbit of $z = 0$ under the following maps:

(a) $g(z) = z - 1$
(b) $f(z) = z + 2$

3. Let $f(z) = az + b, \ a, b \in C$.

(a) Under what conditions does f have fixed points? Then find the fixed points of f if they exist.
(b) Show that if $a \neq 1$, then f is topologically conjugate to a map of the form $z \to cz$.

4. Let $g(z) = az$, with $a = \frac{4}{5}e^{\pi i/3}$.
   (a) Show that the orbit of 1 under g looks like a spiral. Then find the equation of this spiral.
   (b) Show that if z_1 and z_2 are two points on the spiral that lie on the same ray extending from the origin, then there exists $k \in \mathbb{Z}^+$ such that $g^k(z_1) = z_2$ or $g^k(z_2) = z_1$.
5. Consider the map $Q_{1/4}(z) = z^2 + 1/4$.
   (a) Show that $Q_{1/4}(z)$ has a single fixed point. Then determine its stability.
   (b) Find the repelling 2-cycles of $Q_{1/4}$.
6. Show that $Q_c(z) = z^2 + c$ has an attracting 2-cycle inside the circle with radius 1/4 and center $(-1, 0)$.
7. Let $f(z) = e^{i\theta}z$.
   (a) Show that if θ is a rational multiple of π, then every point in $\mathbb{C}$ is periodic.
   (b) Show that if θ is not a rational multiple of π, then the orbit of $z \in \mathbb{C}$ is dense in the circle with radius $|z|$ and center at the origin.
8. Let $Q_0 : \mathbb{C} \to \mathbb{C}$ be defined by $Q_0(z) = z^2$. If S is the circle of radius 1 and center $(-1, 0)$, find and draw $Q_0^{-1}(S)$ and $Q_0^{-2}(S)$.
9. Consider $Q_2(z) = z^2 + 2$ and the unit circle $S = \{z : |z| = 1\}$. Sketch $Q_2^{-1}(S)$ and $Q_2^{-2}(S)$.
10. Prove Theorem 7.1.
11. (a) Show that the stereographic projection takes a point $(a, b, c) \in S^2\backslash\{N\}$ to the point $z = (a + ib)/(1 - c)$ in the complex plane.
    (b) Show the converse, i.e., that a stereographic projection takes a point $z = x + iy$ in the complex plane to the point
$$\left(\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{x^2+y^2}{x^2+y^2+1}\right).$$
12. Show that the map $T(z) = \frac{(1+2i)z+1}{(1-2i)z+1}$ maps the real axis in the complex plane to the unit circle.

13. Show that the map $g(z) = a_2z^2 + 2a_1z + a_0$, with $a_2 \neq 0$ is topologically conjugate to the map $Q_c(z) = z^2 + c$ through the conjugacy map $h(z) = a_2z + a_1$, provided that $c = -a_1^2 + a_1 + a_2a_0$.
14. Prove that any Möbius transformation may be written as a composition of translations (of the form $z \to z + a$), inversions (of the form $z \to \frac{1}{z}$), and homothetic transformations (of the form $z \to bz$).
15. Assume that $(a-d)^2 + 4bc = 0$ in the Möbius transformation $T = \frac{az+b}{cz+d}$.
    (a) Show that T has a unique fixed point $z^* = a - d$.
    (b) Show that T is (analytically) conjugate to a translation of the form $z \to z + a$.
16. Show that if the Möbius map T has two fixed points, then it is (analytically) conjugate to a unique linear map of the form $z \to bz$.

7.3 The Julia Set

In this section our goal is to study the Julia set, one of the most fascinating and extensively studied objects in the theory of dynamical systems. This famous set was introduced by the French mathematician Gaston Julia (1893–1978) in his masterpiece paper, *Mémoire sur l'iteration des fonctions rationelles* (*J. Math. Pure Appl.*, 4, 1918, 47–245). It is interesting to note that Julia was only 25 years old when he published this monumental work of 199 pages.

We begin our exposition by defining two sets; the Julia set and the filled Julia set.

DEFINITION 7.2 *Let $f : \mathbb{C} \to \mathbb{C}$. Then the* **filled Julia set** *$K(f)$ of the map f is defined as*

$$K(f) = \{z \in \mathbb{C} : O(z) \text{ is bounded}\}.$$

The Julia set $J(f)$ of the map f is defined as the boundary of the filled Julia set $K(f)$. Equivalently, one may define J as the boundary of the escape set

$$E = \left\{z \in \mathbb{C} : |f^n(z)| \to \infty \text{ as } n \to \infty\right\}.$$