14. A population of birds is modeled by the difference equation

$$x(n+1) = \begin{cases} 3.2x(n); & \text{for } 0 \le x(n) \le 1 \\ 0.5x(n) + 2.7; & \text{for } x(n) > 1 \end{cases}$$

where $x(n)$ is the number of birds in year n. Find the equilibrium points and then determine their stability.

1.6 Criteria for Stability

In this section, we will establish some simple but powerful criteria for local stability of fixed points. Fixed (equilibrium) points may be divided into two types: **hyperbolic** and **nonhyperbolic**. A fixed point x^* of a map f is said to be **hyperbolic** if $|f'(x^*)| \neq 1$. Otherwise, it is nonhyperbolic. We will treat the stability of each type separately.

1.6.1 Hyperbolic Fixed Points

The following result is the main tool in detecting local stability.

THEOREM 1.3
Let x^* be a hyperbolic fixed point of a map f, where f is continuously differentiable at x^*. The following statements then hold true:

1. *If $|f'(x^*)| < 1$, then x^* is asymptotically stable.*
2. *If $|f'(x^*)| > 1$, then x^* is unstable.*

PROOF 1. Suppose that $|f'(x^*)| < M < 1$ for some $M > 0$. Then, there is an open interval $I = (x^* - \varepsilon,\ x^* + \varepsilon)$ such that $|f'(x)| \le M < 1$ for all $x \in I$ (Why? Problem 11). By the mean value theorem,[5] for any $x_0 \in I$, there exists c between x_0 and x^* such that

$$|f(x_0) - x^*| = |f(x_0) - f(x^*)| = |f'(c)||x_0 - x^*| \le M|x_0 - x^*| . \quad (1.13)$$

[5]The mean value theorem. If f is continuous on the closed interval $[a, b]$ and is differentiable on the open interval (a, b), then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Since $M < 1$, Inequality (1.13) shows that $f(x_0)$ is closer to x^* than x_0. Consequently, $f(x_0) \in I$. Repeating the above argument on $f(x_0)$ instead of x_0, we can show that

$$|f^2(x_0) - x^*| \le M|f(x_0) - x^*| \le M^2|x_0 - x^*| . \quad (1.14)$$

By mathematical induction, we can show that for all $n \in \mathbb{Z}^+$,

$$|f^n(x_0) - x^*| \le M^n|x_0 - x^*| . \quad (1.15)$$

To prove the stability of x^*, for any $\varepsilon > 0$, we let $\delta = \varepsilon$. Then, $|x_0 - x^*| < \delta$ implies that $|f^n(x_0) - x^*| \le M^n|x_0 - x^*| < \varepsilon$, which establishes stability. Furthermore, from Inequality (1.15) $\lim_{n\to\infty} |f^n(x_0) - x^*| = 0$ and thus $\lim_{n\to\infty} f^n(x_0) = x^*$, which yields asymptotic stability. The proof to part 2 is left to you as Problem 13. ∎

The following examples illustrate the applicability of the above theorem.

Example 1.6
Consider the quadratic map $Q_\lambda(x) = 1 - \lambda x^2$ defined on the interval $[-1, 1]$, where $\lambda \in (0, 2]$. Find the fixed points of Q_λ and determine their stability. □

SOLUTION To find the fixed points of Q_λ we solve the equation $\lambda x^2 + x - 1 = 0$. There are two fixed points:

$$x_1^* = \frac{-1 - \sqrt{1+4\lambda}}{2\lambda} \text{ and } x_2^* = \frac{-1 + \sqrt{1+4\lambda}}{2\lambda} .$$

Observe that $Q'_\lambda(x) = -2\lambda x$. Thus, $|Q'_\lambda(x_1^*)| = 1 + \sqrt{1+4\lambda} > 1$, and hence, x_1^* is unstable for all $\lambda \in (0, 2]$. Furthermore, $|Q'_\lambda(x_2^*)| = \sqrt{1+4\lambda} - 1 < 1$ if and only if $\sqrt{1+4\lambda} < 2$. Solving the latter inequality for λ, we obtain $\lambda < \frac{3}{4}$. This implies by Theorem 1.3 that the fixed point x_2^* is asymptotically stable if $0 < \lambda < \frac{3}{4}$ and unstable if $\lambda > \frac{3}{4}$ (see Fig. 1.13). When $\lambda = \frac{3}{4}$, $Q'_\lambda(x_2^*) = -1$. This case will be treated in Section 1.6.2. ∎

Example 1.7
(Raphson-Newton's Method). Raphson-Newton's method is one of the simplest and oldest numerical methods for finding the roots of the equation

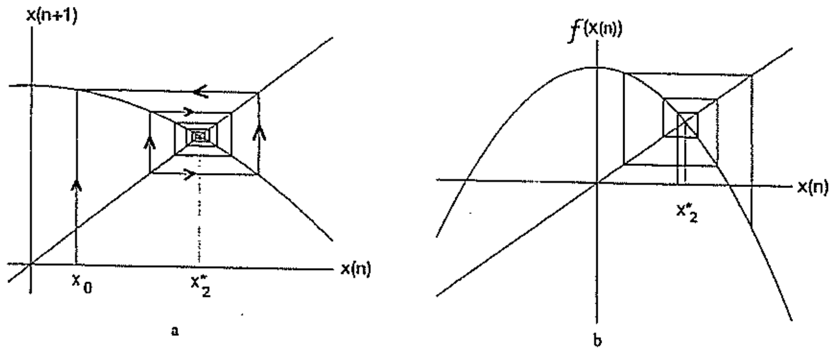

FIGURE 1.13

(a) $\lambda = \frac{1}{2}$, x_2^* is asymptotically stable while (b) $\lambda = \frac{3}{2}$, x_2^* is unstable.

$g(x) = 0$. The Newton algorithm for finding a zero r of $g(x)$ is given by the difference equation

$$x(n+1) = x(n) - \frac{g(x(n))}{g'(x(n))}. \qquad (1.16)$$

where $x(0) = x_0$ is our initial guess of the root r. Equation (1.16) is of the form of Eq. (1.13) with

$$f_N(x) = x - \frac{g(x)}{g'(x)} \qquad (1.17)$$

where f_N is called Newton's function. □

We observe first if r is a root of $g(x)$, i.e., $g(r) = 0$, then from Eq. (1.17) we have $f_N(r) = r$ and thus r is a fixed point of f_N (assuming that $g'(r) \neq 0$). On the other hand, if x^* is a fixed point of f_N, then from Eq. (1.17) again we get $\frac{g(x^*)}{g'(x)} = 0$. This implies that $g(x^*) = 0$, i.e., x^* is a zero of $g(x)$. Now, starting with a point x_0 close to a root r of $g(x) = 0$, then Algorithm (1.16) gives the next approximation $x(1)$ of the root r. By applying the algorithm repeatedly, we obtain the sequence of approximations

$$x_0 = x(0),\ x(1),\ x(2), \ldots,\ x(n), \ldots$$

(see Fig. 1.14). The question is whether or not this sequence converges to the root r. In other words, we need to check the asymptotic stability of the fixed point $x^* = r$ of f_N. To do so, we evaluate $f_N'(r)$ and then use Theorem 1.3,

$$|f_N'(r)| = \left|1 - \frac{[g'(r)]^2 - g(r)g''(r)}{[g'(r)]^2}\right| = 0, \quad \text{since } g(r) = 0\,.$$

Hence, by Theorem 1.3, $\lim_{n\to\infty} x(n) = r$, provided that x_0 is sufficiently close to r.

For $g(x) = x^2 - 1$, we have two zero's -1, 1. In this case, Newton's function is given by $f_N(x) = x - \frac{x^2-1}{2x} = \frac{x^2+1}{2x}$. The cobweb diagram of f_N shows that Newton's algorithm converges quickly to both roots (see Fig. 1.15).

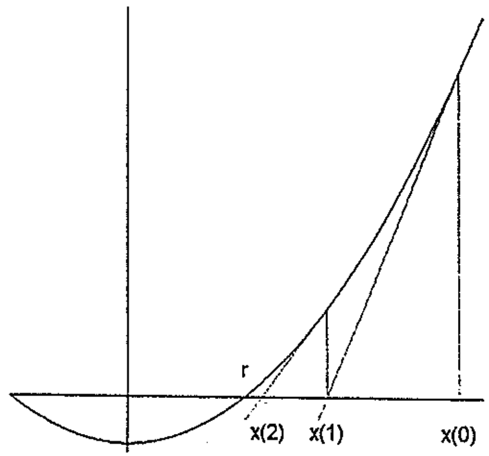

FIGURE 1.14
Newton's method for $g(x) = x^2 - 1$.

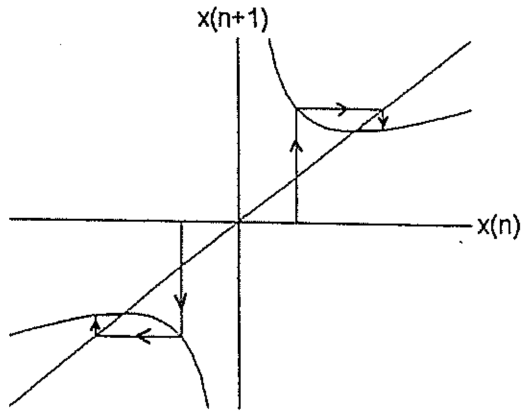

FIGURE 1.15
Cobweb diagram for Newton's function f_N when $g(x) = x^2 - 1$.

1.6.2 Nonhyperbolic Fixed Points

The stability criteria for nonhyperbolic fixed points are more involved. They will be summarized in the next two results, the first of which treats the case when $f'(x^*) = 1$ and the second for $f'(x^*) = -1$.

THEOREM 1.4
Let x^* be a fixed point of a map f such that $f'(x^*) = 1$. If $f'''(x^*) \neq 0$ and continuous, then the following statements hold:

1. *If $f''(x^*) \neq 0$, then x^* is unstable.*
2. *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.*
3. *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable.*

PROOF 1. Assume that $f'(x^*) = 1$ and $f''(x^*) \neq 0$.
Then, the curve $y = f(x)$ is either concave upward ($f''(x^*) > 0$) or concave downward ($f''(x^*) < 0$), as shown in Fig. 1.16(a) and (b). Now, if $f''(x^*) > 0$, then $f'(x)$ is increasing in a small interval containing x^*. Hence, $f'(x) > 1$ for all $x \in (x^*, x^* + \delta)$, for some small $\delta > 0$ [see Fig. 1.16(a)]. Using the same proof as in Theorem 1.3, we conclude that x^* is unstable. Similarly, if $f''(x^*) < 0$ then $f'(x)$ is decreasing in a small neighborhood of x^*. Therefore, $f'(x) > 1$ for all $x \in (x^* - \delta, x^*)$, for some small $\delta > 0$, and again we conclude that x^* is unstable [see Fig. 1.16(b)]. Proofs of parts 2 and 3 are left to you as Problem 14. ∎

The preceding theorem may be used to establish stability criteria for the case when $f'(x^*) = -1$. But before doing so, we need to introduce the notion of the **Schwarzian derivative.**

DEFINITION 1.3 *(The Schwarzian derivative). Sf of a function f is defined by*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2. \tag{1.18}$$

THEOREM 1.5
Let x^* be a fixed point of a map f such that $f'(x^*) = -1$. If $f'''(x^*)$ is continuous, then the following statements hold:

1. *If $Sf(x^*) < 0$, then x^* is asymptotically stable.*

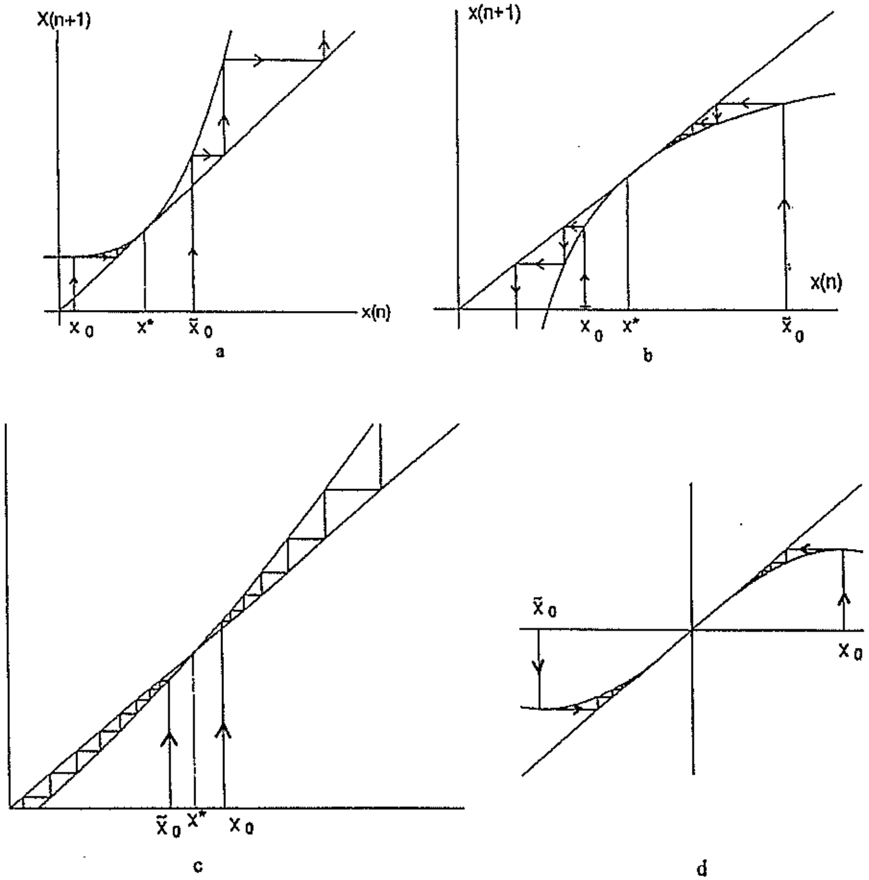

FIGURE 1.16
(a) $f'(x^*) = 1$, $f''(x^*) > 0$, unstable fixed point, semi-stable from the left.
(b) $f'(x^*) = 1$, $f''(x^*) < 0$, unstable fixed point, semi-stable from the right.
(c) $f'(x^*) = 1$, $f''(x^*) = 0$, $f'''(x^*) > 0$, unstable fixed point.
(d) $f'(x^*) = 1$, $f''(x^*) = 0$, $f'''(x^*) < 0$, asymptotically stable fixed point.

2. *If* $Sf(x^*) > 0$, *then* x^* *is unstable.*

PROOF The main idea of the proof is to create an associated function g with the property that $g'(x^*) = 1$, so that we can use Theorem 1.4. This function is indeed $g = f \circ f = f^2$. Two important facts need to be observed here. First, if x^* is a fixed point of f, then it is also a fixed point of g. Second, if x^* is asymptotically stable (unstable) with respect to g, then it is also asymptotically stable (unstable) with respect to f (Why? Problem 15). By the chain rule:

$$g'(x) = \frac{d}{dx} f(f(x)) = f'(f(x))f'(x)\,. \tag{1.19}$$

Hence,

$$g'(x^*) = [f'(x^*)]^2 = 1$$

and Theorem 1.4 now applies. For this reason we compute $g''(x^*)$. From Eq. (1.19), we have

$$\begin{aligned} g''(x) &= f'(f(x))f''(x) + f''(f(x))[f'(x)]^2 \\ g''(x^*) &= f'(x^*)f''(x^*) + f''(x^*)[f'(x^*)]^2 \\ &= 0 \ \text{(since } f'(x^*) = -1)\,. \end{aligned} \tag{1.20}$$

Computing $g'''(x)$ from Eq. 1.20, we get

$$\begin{aligned} g'''(x^*) &= -2f'''(x^*) - 3[f''(x^*)]^2 \\ &= 2Sf(x^*). \end{aligned} \tag{1.21}$$

Statements 1 and 2 now follow immediately from Theorem 1.4.

REMARK 1.2 Note that if $f'(x^*) = -1$, then the Schwarzian derivative $Sf(x^*)$ reduces to

$$Sf(x^*) = -f'''(x^*) - \frac{3}{2}[f''(x^*)]^2\,. \tag{1.22}$$

We are now ready to give an example of a nonhyperbolic fixed point.

Example 1.8

Consider the quadratic map $Q(x) = x^2 + 3x$ on the interval [-3, 3]. Find the equilibrium points and then determine their stability.

SOLUTION The fixed points of Q are obtained by solving the equation $x^2 + 3x = x$. Thus, there are two fixed points: $x_1^* = 0$ and $x_2^* = -2$. So for x_1^*, $Q'(0) = 3$, which implies by Theorem 1.3 that x_1^* is unstable. For x_2^*, we have $Q'(-2) = -1$, which requires employment of Theorem 1.5. We observe that

$$SQ(-2) = -Q'''(-2) - \frac{3}{2}[Q''(-2)]^2 = -6 < 0\,.$$

Hence, x_2^* is asymptotically stable (see Fig. 1.17).

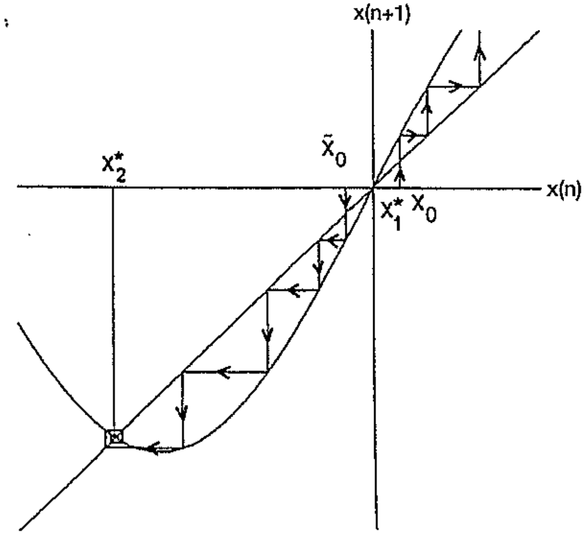

FIGURE 1.17
An asymptotically stable nonhyperbolic fixed point x_2^*.

Exercises - (1.6)

In Problems 1–8, find the equilibrium points and determine their stability.

1. $x(n+1) = x^2(n)$
2. $x(n+1) = \frac{1}{2}x^3(n) + \frac{1}{2}x(n)$