(b) Let $J_n = \{x \in J : G_c^n(x) \in J\}$ and $\tilde{\Lambda} = \bigcap_{n=1}^{\infty} J_n$. Show that if $c < -\frac{(5+2\sqrt{5})}{4}$, then $\tilde{\Lambda}$ is a Cantor set.

13. Prove that the map G_c is chaotic on $\tilde{\Lambda}$ for $c < -\frac{(5+2\sqrt{5})}{4}$.

14. Prove the nested intersection theorem: Let $I_n = [a_n, b_n]$ be a nested sequence of closed intervals, i.e., $I_n \supset I_{n+1}$ for all $n \in \mathbb{Z}^+$, such that if $d_n = |b_n - a_n|$, then $d_n \to 0$ as $n \to \infty$. Then, $\bigcap_{n=0}^{\infty} I_n$ contains exactly one point.

15. Prove Theorem 3.12.

16. Let $f : \mathbb{R} \to \mathbb{R}^1$ be a C^1-map and I_1, I_2 be two disjoint closed bounded intervals. Let $I = I_1 \cup I_2$ and assume that $f(I_i) \supset I$, for $i = 1, 2$. Assume also that $|f'(x)| \geq \lambda > 1$ for all $x \in I \cap f^{-1}(I)$.

    (a) Prove that $\Lambda = \bigcap_{k=0}^{\infty} f^{-k}(I)$ is a Cantor set.

    (b) Define $h : \Lambda \longrightarrow \sum_2^+$ as the itinerary map (3.22). Show that h is a conjugacy map.

    (c) Show that f is chaotic on Λ.

Chapter 4

Stability of Two-Dimensional Maps

Is evolution a matter of survival of the fittest or survival of the most stable?

A. M. Waldrop

4.1 Linear Maps vs. Linear Systems

Recall from linear algebra that a map $L : \mathbb{R}^2 \to \mathbb{R}^2$ is called a linear transformation if

1. $L(U_1 + U_2) = L(U_1) + L(U_2)$ for $U_1, U_2 \in \mathbb{R}^2$
2. $L(\alpha U) = \alpha L(U)$ for $U \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$.

Moreover, it is always possible to represent f (with a given basis for $\mathbb{R}^2$) by a matrix A. A typical example is

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

which may be written in the form

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$L(U) = AU\,, \tag{4.1}$$

where $U = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

By iterating L, we conclude that $L^n(U) = A^n U$. Hence, the orbit of U under f is given by

$$\{U, AU, A^2U, \ldots, A^nU, \ldots\} \tag{4.2}$$

Thus, to compute the orbit of U, it suffices to compute A^nU for $n \in \mathbb{Z}^+$.

Another way of looking at the same problem is by considering the following two-dimensional system of difference equations

$$\begin{aligned} x(n+1) &= ax(n) + by(n) \\ y(n+1) &= cx(n) + dy(n), \end{aligned} \tag{4.3}$$

or

$$U(n+1) = AU(n). \tag{4.4}$$

By iteration, one may show that the solution of Eq. (4.4) is given by

$$U(n) = A^nU(0). \tag{4.5}$$

So, if we let $U_0 = U(0)$, then $L^n(U_0) = U(n)$.

The form of Eq. (4.3) is more convenient when we are considering applications in biology, engineering, economics, and so forth. For example, $x(n)$ and $y(n)$ may represent the population sizes at time period n of two competitive cooperative species, or preys and predators.

In the next section, we will develop the necessary machinery to compute A^n for any matrix of order two. The general theory may be found in [22, 23, 37].

4.2 Computing A^n

Consider a matrix $A = (a_{ij})$ of order 2×2. Then, $p(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of A and its zeros are called the eigenvalues of A. Associated with each eigenvalue λ of A a nonzero eigenvector $V \in \mathbb{R}^2$ with $AV = \lambda V$.

Example 4.1

Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}. \quad \square$$

SOLUTION First we find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ or

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

which is

$$\lambda^2 - 6\lambda + 5 = 0.$$

Hence, $\lambda_1 = 1$ and $\lambda_2 = 5$. To find the corresponding eigenvector V_1, we solve the vector equation $AV_1 = \lambda V_1$ or $(A - \lambda_1 I)V_1 = 0$.

For $\lambda_1 = 1$, we have

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $v_{11} + 3v_{21} = 0$. Thus, $v_{11} = -3v_{21}$. So, if we let $v_{21} = 1$, then $v_{11} = -3$. It follows that the eigenvector V_1 corresponding to λ_1 is given by $V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 5$, the corresponding eigenvector may be found by solving the equation $(A - \lambda_2 I)V_2 = 0$. This yields

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $-3v_{12} + 3v_{22} = 0$ or $v_{12} = v_{22}$. It is then appropriate to let $v_{12} = v_{22} = 1$ and hence $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. ∎

To find the general form for A^n for a general matrix A is a formidable task even for a 2×2 matrix such as in Example 4.1. Fortunately, however, we may be able to transform a matrix A to another simpler matrix B whose nth power B^n can easily be computed. The essence of this process is captured in the following definition.

DEFINITION 4.1 *The matrices A and B are said to be similar if there exists a nonsingular*[1] *matrix P such that*

$$P^{-1}AP = B.$$

[1] A matrix P is said to be nonsingular if its inverse P^{-1} exists. This is equivalent to saying that $\det P \neq 0$, where det denotes determinant.

We note here that the relation "similarity" between matrices is an equivalence relation, i.e.,

1. A is similar to A.
2. If A is similar to B then B is similar to A.
3. If A is similar to B and B is similar to C, then A is similar to C.

The most important feature of similar matrices, however, is that they possess the same eigenvalues.

THEOREM 4.1
Let A and B be two similar matrices. Then A and B have the same eigenvalues.

PROOF Suppose that $P^{-1}AP = B$ or $A = PBP^{-1}$. Let λ be an eigenvalue of A and V be the corresponding eigenvector. Then, $\lambda V = AV = PBP^{-1}V$. Hence, $B(P^{-1}V) = \lambda(P^{-1}V)$. Consequently, λ is an eigenvalue of B with $P^{-1}V$ as the corresponding eigenvector.

The notion of similarity between matrices corresponds to linear conjugacy, which we have encountered in Chapter 3. In other words, two linear maps are conjugate if their corresponding matrix representations are similar. Thus, the linear maps L_1, L_2 on $\mathbb{R}^2$ are linearly conjugate if there exists an invertible map h such that

$$L_1 \circ h = h \circ L_2$$

or

$$h^{-1} \circ L_1 \circ h = L_2 . \quad \blacksquare$$

The next theorem tells us that there are three simple "canonical" forms for 2×2 matrices.

THEOREM 4.2
Let A be a 2×2 real matrix. Then A is similar to one of the following matrices:

1. $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
2. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$
3. $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

PROOF Suppose that the eigenvalues λ_1 and λ_2 are real. Then, we have two cases to consider. The first case is where $\lambda_1 \neq \lambda_2$. In this case, we may easily show that the corresponding eigenvectors V_1 and V_2 are linearly independent (Problem 10). Hence, the matrix $P = (V_1, V_2)$, i.e., the matrix P whose columns are these eigenvectors, is nonsingular. Let $P^{-1}AP = J = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then,

$$AP = PJ . \tag{4.6}$$

Comparing both sides of Eq. (4.6), we obtain

$$AV_1 = eV_1 + gV_2 .$$

Hence,

$$\lambda_1 V_1 = eV_1 + gV_2 .$$

Thus, $e = \lambda_1$ and $g = 0$.

Similarly, one may show that $f = 0$ and $h = \lambda_2$. Consequently, J is a diagonal matrix of the form (a). $\blacksquare$

The second case is where $\lambda_1 = \lambda_2 = \lambda$. There are two subcases to consider here. The first subcase occurs if we are able to find two linearly independent eigenvectors V_1 and V_2 corresponding to the eigenvalue λ. This subcase is then reduced to the preceding case. We note here that this scenario happens when $(A - \lambda I)V = 0$ for all $V \in \mathbb{R}^2$. In particular, one may let $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which are clearly linearly independent.

The second subcase occurs when there exists a nonzero vector $V_2 \in \mathbb{R}^2$ such that $(A - \lambda I)V_2 \neq 0$. Equivalently, we are able to find only one eigenvector (not counting multiples) V_1 with $(A - \lambda I)V_1 = 0$. In practice, we find V_2 by solving the equation

$$(A - \lambda I)V_2 = V_1 .$$

The vector V_2 is called a generalized eigenvector of A. Note that $AV_1 = \lambda V_1$ and $AV_2 = \lambda V_2 + V_1$. Now, we let $P = (V_1, V_2)$ and $P^{-1}AP = J$. Then,

$$AP = PJ . \tag{4.7}$$

Comparing both sides of Eq. (4.7) yields

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} . \tag{4.8}$$

The matrix J is in a Jordan form.

Next, we assume that A has a complex eigenvalue $\lambda_1 = \alpha + i\beta$. Since A is assumed to be real, it follows that the second eigenvalue λ_2 is a conjugate of λ_1, that is, $\lambda_2 = \alpha - i\beta$. Let $V = V_1 + iV_2$ be the eigenvector corresponding to λ_1. Then,

$$AV = \lambda_1 V$$
$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2)\,.$$

Hence,

$$AV_1 = \alpha V_1 - \beta V_2$$
$$AV_2 = \beta V_1 + \alpha V_2\,,$$

letting $P = (V_1, V_2)$ we get $P^{-1}AP = J$. Hence,

$$AP = PJ\,. \tag{4.9}$$

Comparison of both sides of Eq. (4.9) yields

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{4.10}$$

Theorem 4.2 gives us a simple method of computing the general form of A^n for any 2×2 real matrix. In the first case, when $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, we have

$$\begin{aligned} A^n &= (PDP^{-1})^n \\ &= PD^nP^{-1} \\ &= P\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}P^{-1}\,. \end{aligned} \tag{4.11}$$

In the second case, when $P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then

$$\begin{aligned} A^n &= PJ^nP^{-1} \\ &= P\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}P^{-1}\,. \end{aligned} \tag{4.12}$$

Equation (4.12) may be easily proved by mathematical induction (Problem 11).

In the third case, we have $P^{-1}AP = J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Let $\omega = \arctan(\beta/\alpha)$. Then $\cos\omega = \alpha/|\lambda_1|$, $\sin\omega = \beta/|\lambda_1|$. Now, we write the matrix J in the form

$$J = |\lambda_1|\begin{pmatrix} \alpha/|\lambda_1| & \beta/|\lambda_1| \\ -\beta/|\lambda_1| & \alpha/|\lambda_1| \end{pmatrix} = |\lambda_1|\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

By mathematical induction one may show that (Problem 11)

$$J^n = |\lambda_1|^n\begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}. \tag{4.13}$$

and thus

$$A^n = |\lambda_1|^n P\begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}P^{-1}\,. \tag{4.14}$$

Example 4.2

Solve the system of difference equations

$$X(n+1) = AX(n) \tag{4.15}$$

where

$$A = \begin{pmatrix} -4 & 9 \\ -4 & 8 \end{pmatrix},\quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad \square$$

SOLUTION The eigenvalues of A are repeated: $\lambda_1 = \lambda_2 = 2$. The only eigenvector that we are able to find is $V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. To construct P we need to find a generalized eigenvector V_2. This is accomplished by solving the equation $(A - 2I)V_2 = V_1$. Then, V_2 may be taken as any vector $\begin{pmatrix} x \\ y \end{pmatrix}$, with $3y - 2x = 1$. We take $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now if we put $P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, then $P^{-1}AP = J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Thus, the solution of Eq. (4.15) is given by

$$\begin{aligned} X(n) &= PJ^nP^{-1}x(0) \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 2^n\begin{pmatrix} 1 - 3n \\ -2n \end{pmatrix}. \quad \blacksquare \end{aligned}$$

REMARK 4.1 If a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(X_0) = AX_0$, then $f^n(X_0) = A^n X_0 = PJ^nP^{-1}X_0$. In particular, if $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $f^n(X_0) = 2^n \begin{pmatrix} 1-3n \\ -2n \end{pmatrix}$ for all $n \in \mathbb{Z}^+$. ∎

Exercises - (4.1 and 4.2)

In Problems 1–5, find the eigenvalues and eigenvectors of the matrix A and then compute A^n.

1. $A = \begin{pmatrix} -4.5 & 5 \\ -7.5 & 8 \end{pmatrix}$

2. $A = \begin{pmatrix} 4.5 & -1 \\ 2.25 & 1.5 \end{pmatrix}$

3. $A = \begin{pmatrix} 8/3 & 1/3 \\ -4/3 & 4/3 \end{pmatrix}$

4. $A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$

5. $A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$

6. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(X) = AX$ where A is as in Problem 1. Find $L^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

7. Solve the difference equation $X(n+1) = AX(n)$ where A is as in Problem 3 and $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

8. Solve the difference equation $X(n+1) = AX(n)$ where A is as in Problem 4 and $X(0) = X_0$.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(X) = AX$, with A as in Problem 5. Find $f^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

10. Let A be a 2×2 matrix with distinct real eigenvalues. Show that the corresponding eigenvectors of A are linearly independent.

11. (a) If $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, show that $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

    (b) If $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, show that $J^n = |\lambda|^n \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}$, where $|\lambda| = \sqrt{\alpha^2 + \beta^2}$, $\omega = \arctan\left(\frac{\beta}{\alpha}\right)$.

12. Let a matrix A be in the form

$$A = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix}$$

(a) Show that if A has distinct eigenvalues λ_1 and λ_2, then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$.

(b) Show that if A has a repeated eigenvalue λ, then

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$.

(c) Show that if A has complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then

$$P^{-1}AP = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

where $P = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$.

4.3 Fundamental Set of Solutions

Consider the linear system

$$X(n+1) = AX(n), \tag{4.16}$$

where A is a 2×2 matrix. Then, two solutions $X_1(n)$ and $X_2(n)$ of Eq. (4.16) are said to be linearly independent if $X_2(n)$ is not a scaler multiple of $X_1(n)$ for all $n \in \mathbb{Z}^+$. In other words, if $c_1X_1(n) + c_2X_2(n) = 0$ for all $n \in \mathbb{Z}^+$, then $c_1 = c_2 = 0$. A set of two linearly independent solutions $\{X_1(n), X_2(n)\}$ is called a fundamental set of solutions of Eq. (4.16).

DEFINITION 4.2 *Let $\{X_1(n), X_2(n)\}$ be a fundamental set of solutions of Eq. (4.16). Then*

$$X(n) = k_1X_1(n) + K_2X_2(n),\ k_1, k_2 \in \mathbb{R} \tag{4.17}$$

is called a general solution of Eq. (4.16).

Finding $X_1(n)$ and $X_2(n)$ is generally an easy task. We now give an explicit derivation.

In the sequel λ_1, λ_2 denote the eigenvalues of A; V_1, V_2 are the corresponding eigenvectors of A.

CASE 4.1

Suppose that $P^{-1}AP = J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. *Then a general solution may be given by*

$$\begin{aligned} X(n) = A^nX(0) &= PJ^nP^{-1}X(0) \\ &= (V_1,\ V_2)\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \end{aligned}$$

where $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = P^{-1}X(0)$. *Then,*

$$X(n) = k_1\lambda_1^nV_1 + k_2\lambda_2^nV_2\,. \tag{4.18}$$

Here, $X_1(n) = \lambda_1^nV_1$ and $X_2(n) = \lambda_2^nV_2$ constitute a fundamental set of solutions since in this case V_1 and V_2 are linearly independent eigenvectors. Note that one may check directly that $\lambda_1^nV_1$ and $\lambda_2^nV_2$ are indeed solutions of Eq. (4.16) (Problem 13a).

CASE 4.2

Suppose that $P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. *Then, a general solution may be given by*

$$\begin{aligned} X(n) &= PJ^nP^{-1}Y(0) \\ &= (V_1,\ V_2)\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &= k_1\lambda^nV_1 + k_2(n\lambda^{n-1}V_1 + \lambda^nV_2) \end{aligned} \tag{4.19}$$

Hence, $X_1(n) = \lambda^nV_1$ and $X_2(n) = \lambda^nV_2 + n\lambda^{n-1}V_1$ constitute a fundamental set of solutions of Eq. (4.16) (Problem 13b).

CASE 4.3

Suppose that $P^{-1}AP = J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. *If $\omega = \arctan(\beta/\alpha)$, then the general solution may be given by*

$$\begin{aligned} X(n) &= PJ^nP^{-1}x(0) \\ &= (V_1V_2)|\lambda_1|^n\begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &= |\lambda_1|^n[k_1\cos n\omega + k_2\sin n\omega)V_1 \\ &\quad +(-k_1\sin n\omega + k_2\cos n\omega)V_2]. \end{aligned} \tag{4.20}$$

Hence, $X_1(n) = |\lambda_1|^n[(k_1\cos n\omega)V_1 - (k_1\sin(n\omega))V_2]$ and $X_2(n) = (|\lambda_1|^n[(k_2\sin(n\omega)V_1 + (k_2\cos(n\omega)]V_2$ constitute a fundamental set of solutions (Problem 13c).

Example 4.3
Solve the system of difference equations

$$X(n+1) = AX(n),\ X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}. \quad \square$$

SOLUTION The eigenvalues of A are $\lambda_1 = -2+3i$ and $\lambda_2 = -2-3i$. The corresponding eigenvectors are $V = \begin{pmatrix} -1 \\ i \end{pmatrix}$ and $\overline{V} = \begin{pmatrix} -1 \\ -i \end{pmatrix}$, respectively.

This time, we take a short cut and use Eq. (4.20). The vectors V_1 and V_2 referred to in this formula are the real part of V, $V_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and the imaginary part of V, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now, $|\lambda_1| = \sqrt{13}$, $\omega = \arctan(\frac{-3}{2}) \approx 123.69°$. Thus,

$$X(n) = (13)^{n/2}[(k_1 \cos n\omega + k_2 \sin n\omega)\begin{pmatrix} -1 \\ 0 \end{pmatrix} + (-k_1 \sin n\omega + k_2 \cos n\omega)\begin{pmatrix} 0 \\ 1 \end{pmatrix}].$$

$$X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = k_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, $k_1 = 1$, $k_2 = 2$. Thus,

$$\begin{aligned} X(n) &= (13)^{n/2}[(\cos n\omega + 2\sin n\omega)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-\sin n\omega + 2\cos n\omega)\begin{pmatrix} 0 \\ 1 \end{pmatrix}] \\ &= (13)^{n/2}\begin{pmatrix} -\cos n\omega - 2\sin n\omega \\ -\sin n\omega + 2\cos n\omega \end{pmatrix}. \end{aligned}$$ ■

4.4 Second-Order Difference Equations

A second-order difference equation with constant coefficients is a scalar equation of the form

$$u(n+2) + p_1 u(n+1) + p_2 u(n) = 0 \tag{4.21}$$

Although one may solve this equation directly, it is sometimes beneficial to convert it to a two-dimensional system. The trick is to let $u(n) = x_1(n)$ and $u(n+1) = x_2(n)$.

Then we have

$$\begin{aligned} x_1(n+1) &= x_2(n) \\ x_2(n+1) &= -p_2 x_1(n) - p_1 x_2(n) \end{aligned}$$

which is of the form

$$X(n+1) = AX(n) \tag{4.22}$$

where

$$X(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix}.$$

The characteristic equation of A is given by

$$\lambda^2 + p_1\lambda + p_2 = 0. \tag{4.23}$$

Observe that we may obtain the characteristic Eq. (4.23) by letting $u(n) = \lambda^n$ in Eq. (4.21). Thus, if λ_1 and λ_2 are the roots of Eq. (4.23), then $u_1(n) = \lambda_1^n$ and $u_2(n) = \lambda_2^n$ are solutions of Eq. (4.21).

Using Eqs. (4.18), (4.19), and (4.20), we can make the following conclusions:

1. If $\lambda_1 \neq \lambda_2$ and both are real, then the general solution of Eq. (4.21) is given by
$$u(n) = c_1\lambda_1^n + c_2\lambda_2^n, \tag{4.24}$$
2. If $\lambda_1 = \lambda_2 = \lambda$, then the general solution of Eq. (4.21) is given by
$$u(n) = c_1\lambda^n + c_2 n\lambda^n, \tag{4.25}$$
3. If $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, then the general solution of Eq. (4.21) is given by
$$u(n) = |\lambda_1|^n(c_1 \cos n\omega + c_2 \sin n\omega), \tag{4.26}$$
where $\omega = \arctan(\beta/\alpha)$.

Example 4.4

Solve the second-order difference equation

$$x(n+2) + 6x(n+1) + 9x(n) = 0, \ x(0) = 1, \ x(1) = 0.$$ □

SOLUTION The characteristic equation associated with the equation is given by $\lambda^2 + 6\lambda + 9 = 0$.

Hence, the characteristic roots are $\lambda_1 = \lambda_2 = -3$. The general solution is given by

$$\begin{aligned} x(n) &= 9(-3)^n + c_2 n(-3)^n \\ x(0) &= 1 = c_1 \\ x(1) &= 0 = -3c_1 - 3c_2. \end{aligned}$$

Thus, $c_2 = -1$ and, consequently,

$$\begin{aligned} x(n) &= (-3)^n - n(-3)^n \\ &= (-3)^n(1-n) \end{aligned}$$ ■