§1.4 HYPERBOLICITY

Simple maps like $id(x) = x$ and $f(x) = -x$ are, unfortunately, atypical among dynamical systems. There are many reasons why this is so, but perhaps the most unusual feature of these maps is the fact that all points are periodic under iteration of these maps. Most maps do not have this type of behavior. Periodic points tend to be more spread out on the line. In this section we will introduce one of the main themes of this book, hyperbolicity. Maps with hyperbolic periodic points are the ones that occur typically in many dynamical systems and, moreover, they provide the simplest types of periodic behavior to analyze.

Definition 4.1. Let p be a periodic point of prime period n. The point p is hyperbolic if $|(f^n)'(p)| \neq 1$. The number $(f^n)'(p)$ is called the multiplier of the periodic point.

Example 4.2. Consider the diffeomorphism $f(x) = \frac{1}{2}(x^3 + x)$. There are 3 fixed points: $x = 0, 1$, and -1. Note that $f'(0) = 1/2$ and $f'(\pm 1) = 2$. Hence each fixed point is hyperbolic. The graph and phase portrait of $f(x)$ are depicted in Fig. 4.1.

Example 4.3. Let $f(x) = -\frac{1}{2}(x^3 + x)$. 0 is a hyperbolic fixed point, with $f'(0) = -\frac{1}{2}$. The points ± 1 now lie on a periodic orbit of period 2. We compute $(f^2)'(\pm 1) = f'(1) \cdot f'(-1) = 4$ by the chain rule. Hence this periodic point is hyperbolic, and the phase portrait is depicted in Fig. 4.2. Note that points in the interval $(-1, 1)$ spiral toward 0 and away from ± 1.

We observe that, in the above two examples, we have $|f'(0)| < 1$ and that points close to 0 are forward asymptotic to 0. This situation occurs often:

Proposition 4.4. *Let p be a hyperbolic fixed point with $|f'(p)| < 1$. Then there is an open interval U about p such that if $x \in U$, then*

$$\lim_{n \to \infty} f^n(x) = p.$$

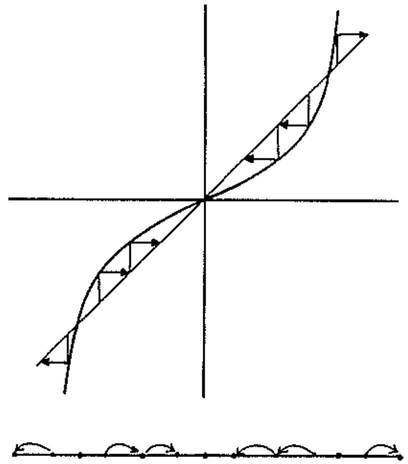

Fig. 4.1. The graph and phase portraits of $f(x) = \frac{1}{2}(x^3 + x)$.

Proof. Since f is C^1, there is $\epsilon > 0$ such that $|f'(x)| < A < 1$ for $x \in [p - \epsilon, p + \epsilon]$. By the Mean Value Theorem

$$|f(x) - p| = |f(x) - f(p)| \leq A|x - p| < |x - p| \leq \epsilon.$$

Hence $f(x)$ is contained in $[p - \epsilon, p + \epsilon]$ and, in fact, is closer to p than x is. Via the same argument

$$|f^n(x) - p| \leq A^n |x - p|$$

so that $f^n(x) \to p$ as $n \to \infty$.

q.e.d.

Remarks.

1. It follows that the interval $[p - \epsilon, p + \epsilon]$ is contained in the stable set associated to p, $W^s(p)$.

2. A similar result is true for hyperbolic periodic points of period n. In this case, we get an open interval U about p which is mapped inside itself by f^n. Of course, the assumption in this case is that $|(f^n)'(p)| < 1$.

Definition 4.5. Let p be a hyperbolic periodic point of period n with $|(f^n)'(p)| < 1$. The point p is called an attracting periodic point (an attractor) or a sink.

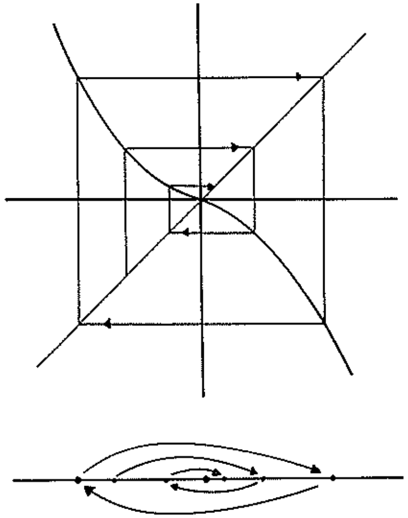

Fig. 4.2. The graph and phase portraits of $f(x) = -\frac{1}{2}(x^3 + x)$.

Attracting periodic points of period n thus have neighborhoods which are mapped inside themselves by f^n. Such a neighborhood is called the *local stable set* and is denoted by W^s_{loc}. We may actually distinguish three different types of attracting fixed points, namely those where $f'(p) = 0$, $0 < f'(p) < 1$, and $-1 < f'(p) < 0$. The behavior near these types of fixed points is illustrated in Fig. 4.3.

The behavior of a map near periodic points where the derivative is larger than one in absolute value is quite different from that of sinks.

Proposition 4.6. *Let p be a hyperbolic fixed point with $|f'(p)| > 1$. Then there is an open interval U of p such that, if $x \in U$, $x \neq p$, then there exists $k > 0$ such that $f^k(x) \notin U$.*

The proof is similar to the proof of the preceding proposition and is therefore left as an exercise. Graphically, the result is quite clear; see Fig. 4.4.

Definition 4.7. A fixed point p with $|f'(p)| > 1$ is called a repelling fixed point (a repellor) or source. The neighborhood described in the Proposition is called the local unstable set and denoted W^u_{loc}.

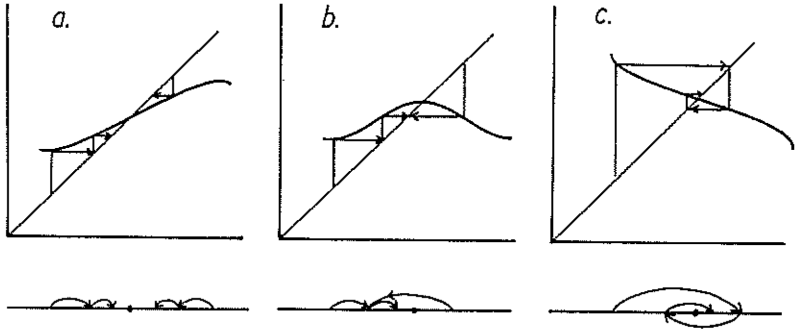

Fig. 4.3. The phase portraits near an attracting fixed point p. in case a. $0 < f'(p) < 1$, b. $f'(p) = 0$, c. $-1 < f'(p) < 0$.

We remark that periodic points of period n exhibit similar behavior when $|(f^n)'(p)| > 1$.

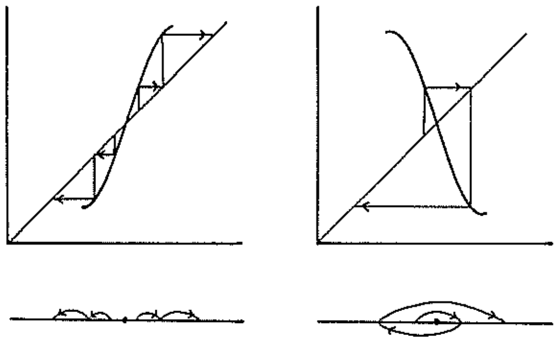

Fig. 4.4. The phase portraits near a repelling fixed point.

Hyperbolic periodic points therefore have local behavior which is governed by the derivative at the periodic point. This is not true when the point is indifferent or non-hyperbolic, as the following example shows.

Example 4.8. Each of the maps in Fig. 4.5 satisfy $f(0) = 0$ and $f'(0) = 1$, but each have vastly different phase portraits near 0. In a., the map $f(x) = x + x^3$ has a *weakly* repelling fixed point at 0. In b., the map $f(x) = x - x^3$

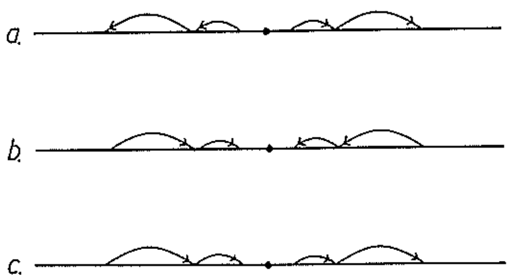

Fig. 4.5. The phase portraits of a. $f(x) = x + x^3$, b. $f(x) = x - x^3$, c. $f(x) = x + x^2$.

has a *weakly* attracting fixed point at 0. In c., the map $f(x) = x + x^2$ is weakly repelling from the right but weakly attracting from the left.

Most maps have only hyperbolic periodic points, as we shall see later. However, non-hyperbolic periodic points often occur in families of maps. When this happens, the periodic point structure often undergoes a *bifurcation*. We will deal with bifurcation theory more extensively later, but for now we give several examples.

Example 4.9. Consider the family of quadratic functions $Q_c(x) = x^2 + c$, where c is a parameter. The graphs of Q_c assume three different positions relative to the diagonal depending upon whether $c > 1/4$, $c = 1/4$, or $c < 1/4$. See Fig. 4.6. Note that Q_c has no fixed points for $c > 1/4$. When $c = 1/4$, Q_c has a unique non-hyperbolic fixed point at $x = 1/2$. And when $c < 1/4$, Q_c has a pair of fixed points, one attracting and one repelling. Thus the phase portrait of Q_c changes as c decreases through 1/4. This change is an example of a bifurcation.

Example 4.10. Let $F_\mu(x) = \mu x(1 - x)$ with $\mu > 1$. F_μ has two fixed points: one at 0 and the other at $p_\mu = (\mu - 1)/\mu$. Note that $F_\mu'(0) = \mu$ and $F_\mu'(p_\mu) = 2 - \mu$. Hence 0 is a repelling fixed point for $\mu > 1$ and p_μ is attracting for $1 < \mu < 3$. When $\mu = 3$, $F_\mu'(p_\mu) = -1$. We sketch the graphs of F_μ^2 for μ near 3. See Fig. 4.7. Note that 2 new fixed points for F_μ^2 appear as μ increases through 3. These are new periodic points of period 2. Another bifurcation has occurred: this time we have a change in $\mathrm{Per}_2(F_\mu)$.

This quadratic family actually exhibits many of the phenomena that are crucial in the general theory. The next section is devoted entirely to this function.

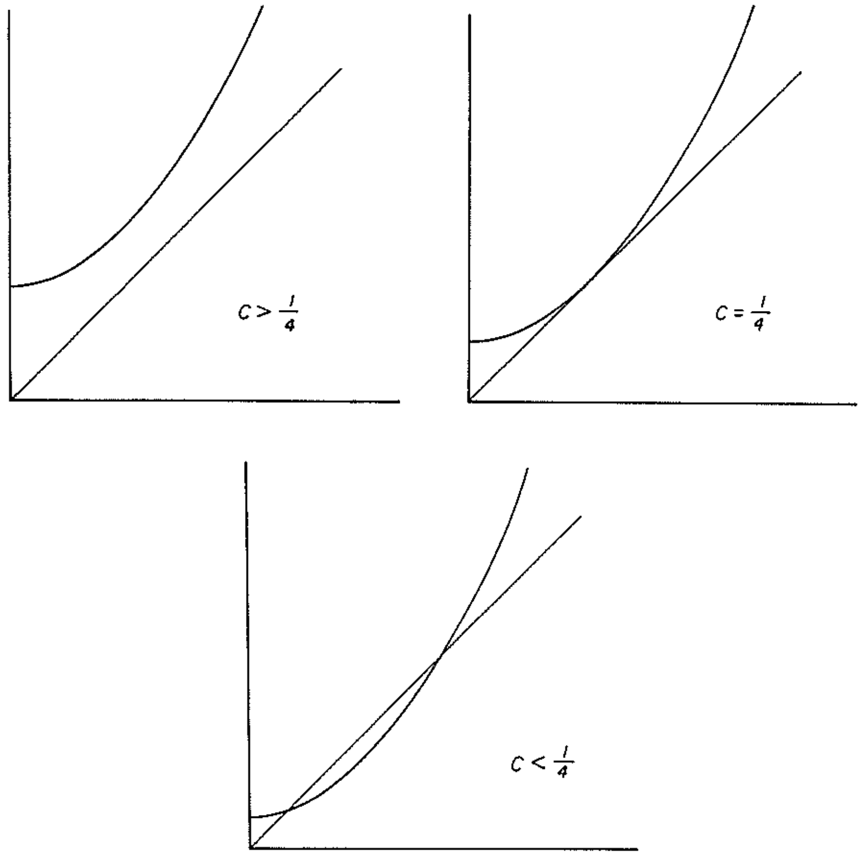

Fig. 4.6. The graphs of $Q_c(x) = x^2 + c$ for $c > 1/4$, $c = 1/4$, and $c < 1/4$.

Exercises

1. Find all periodic points for each of the following maps and classify then as attracting, repelling, or neither. Sketch the phase portraits.

a. $f(x) = x - x^2$

b. $f(x) = 2(x - x^2)$

c. $f(x) = x^3 - \frac{1}{9}x$

d. $f(x) = x^3 - x$

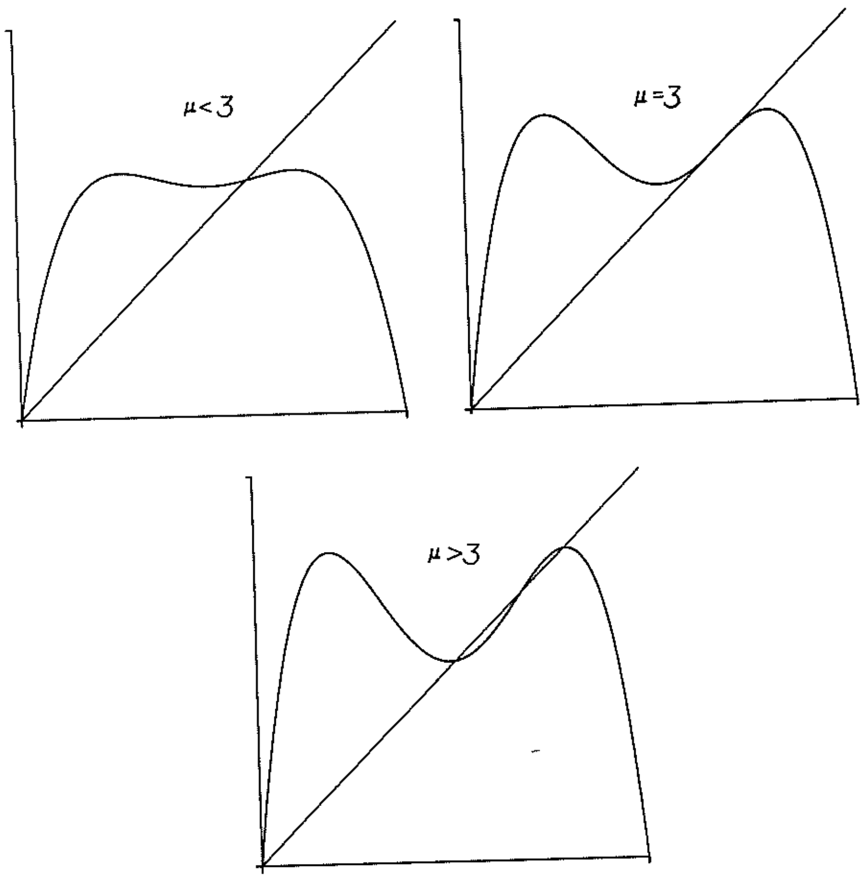

Fig. 4.7. The graphs of $F_\mu^2(x)$ where $F_\mu(x) = \mu x(1-x)$ for $\mu < 3, \mu = 3$, and $\mu > 3$.

e. $S(x) = \frac{1}{2}\sin(x)$

f. $S(x) = \sin(x)$

g. $E(x) = e^{x-1}$

h. $E(x) = e^x$

i. $A(x) = \arctan x$

j. $A(x) = \frac{\pi}{4}\arctan x$

k. $A(x) = -\frac{\pi}{4}\arctan x$

2. Discuss the bifurcations which occur in the following families of maps for the indicated parameter value

a. $S_\lambda(x) = \lambda \sin x, \quad \lambda = 1$

b. $E_\lambda(x) = \lambda e^x, \quad \lambda = 1/e$

c. $E_\lambda(x) = \lambda e^x, \quad \lambda = -e$

d. $Q_c(x) = x^2 + c, \quad c = -3/4$

e. $F_\mu(x) = \mu x(1-x), \quad \mu = 1$

f. $A_\lambda(x) = \lambda \arctan x, \quad \lambda = 1$

g. $A_\lambda(x) = \lambda \arctan x, \quad \lambda = -1$

3. Suppose f is a diffeomorphism. Prove that all hyperbolic periodic points are isolated.

4. Show via an example that hyperbolic periodic points need not be isolated.

5. Find an example of a C^1 diffeomorphism with a non-hyperbolic fixed point which is an accumulation point of other hyperbolic fixed points.

6. Discuss the dynamics of the family $f_\alpha(x) = x^3 - \alpha x$ for $-\infty < \alpha \leq 1$. Find all parameter values where bifurcations occur. Describe how the phase portrait of f_α changes at these points.

7. Consider the linear maps $f_k(x) = kx$. Show that there are four open sets of parameters for which the phase portraits of f_k are similar. The exceptional cases are $k = 0, \pm 1$.

§1.5 AN EXAMPLE: THE QUADRATIC FAMILY

In this section, we will continue the discussion of the quadratic family $F_\mu(x) = \mu x(1-x)$. Actually, we will return to this example repeatedly throughout the remainder of this chapter, since it illustrates many of the most important phenomena that occur in dynamical systems.

The map F is piecewise linear on $[1/3, 2/3]$ and $[2/3, 1]$. Moreover, $F(\frac{2}{3}) = 0$, $F(1) = \frac{1}{3}$, and F is continuous.

Note that F maps $[0, \frac{1}{3}]$ into $[\frac{2}{3}, 1]$ and vice versa. Also note that if $x \in [\frac{1}{3}, \frac{2}{3}]$ and x is not the fixed point, then there exists n so that $F^n(x) \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. This implies that there are no other F-periodic points in $(\frac{1}{3}, \frac{2}{3})$. Exercise 7 shows that if x is a periodic point of period n for f, then $x/3$ is periodic of period $2n$ for F. On the other hand, if y is F-periodic then either y or $F(y)$ lies in $[0, \frac{1}{3}]$ and Exercise 9 shows that $3y$ or $3F(y)$ is f-periodic. Thus to produce a map with period 10 but not period 6, we need only double the graph of a function with period 5 but not period 3.

As a final remark, we must emphasize that Sarkovskii's Theorem is very definitely only a one-dimensional result. There is no higher dimensional analogue of this result. In fact, the Theorem does not even hold on the circle. For example, the map which rotates all points on the circle by 120° makes all points periodic with period three. There are no other periods whatsoever.

Exercises

1. Suppose $A_0, A_1, \ldots, A_n$ are closed intervals and $f(A_i) \supset A_{i+1}$ for $i = 0, \ldots, n-1$. Prove that there exists a point $x \in A_0$ such that $f^i(x) \in A_i$ for each i.

2. Prove that if f has period $p \cdot 2^m$ with p odd, then f has period $q \cdot 2^m$ with q odd, $q > p$.

3. Prove that if f has period $p \cdot 2^m$ with p odd, then f has period 2^ℓ, $\ell \leq m$.

4. Prove that if f has period $p \cdot 2^m$ with p odd, then f has period $q \cdot 2^m$ with q even.

5. Construct a piecewise linear map with period $2n + 1$.

6. Give a formula for $F(x)$ in terms of $f(x)$, where $F(x)$ is the double of $f(x)$.

7. Prove that $F(x)$, the double of $f(x)$, has a periodic point of period $2n$ at $x/3$ iff x has f-period n.

8. Construct a map that has periodic points of period 2^j for $j < \ell$ but not period 2^ℓ.

9. Prove that if $F(x)$, the double of $f(x)$, has a periodic point p that is not fixed, then either p or $F(p)$ lies in $[0, \frac{1}{3}]$. Prove that, in this case, either $3p$ or $3F(p)$ is a periodic point for f.

§1.11 THE SCHWARZIAN DERIVATIVE

In this section, we describe a tool first introduced into the study of one-dimensional dynamical systems by Singer in 1978. This is the Schwarzian derivative. Actually, the Schwarzian derivative plays an important role in complex analysis, where it is used as a criterion for a complex function to be a linear fractional transformation. In one-dimensional dynamics, the Schwarzian derivative is a valuable tool for a number of reasons. In this section, we will show how it may be used to establish an upper bound on the number of attracting periodic orbits that certain maps may have. We will also use it to prove that other maps have an entire interval on which the map is chaotic. Later, in §§ 17–19, the Schwarzian derivative will play an important role in our discussion of how families of maps like the quadratic family make the transition from simple to chaotic dynamics.

Definition 11.1 The Schwarzian derivative of a function f at x is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

For example, if $F_\mu(x) = \mu x(1-x)$ is our quadratic model mapping, then $SF_\mu(x) = -6/(1-2x)^2$, so that $SF_\mu(x) < 0$ for all x (even $x = 1/2$, the critical point, at which $SF_\mu(x) = -\infty$).

For us, functions with negative Schwarzian derivative will be most important. Besides the quadratic map, many other functions have negative Schwarzian derivatives. For example, $S(e^x) = -1/2$ and $S(\sin x) = -1 - \frac{3}{2}(\tan^2 x) < 0$. Many polynomials have this property, as the following proposition shows.

Proposition 11.2. *Let $P(x)$ be a polynomial. If all of the roots of $P'(x)$ are real and distinct, then $SP < 0$.*

Proof. Suppose

$$P'(x) = \prod_{i=1}^{N}(x - a_i)$$

with the a_i distinct and real. Then we have

$$P''(x) = \sum_{j=1}^{N} \frac{P'(x)}{x - a_j} = \sum_{j=1}^{N} \frac{\prod_{i=1}^{N}(x - a_i)}{x - a_j}$$

$$P'''(x) = \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\prod_{i=1}^{N}(x - a_i)}{(x - a_j)(x - a_k)}.$$

Hence we have

$$SP(x) = \sum_{j \neq k} \frac{1}{(x - a_j)(x - a_k)} - \frac{3}{2}\Big(\sum_{j=1}^{N} \frac{1}{x - a_j}\Big)^2$$

$$= -\frac{1}{2} \sum_{j=1}^{N} \Big(\frac{1}{x - a_j}\Big)^2 - \Big(\sum_{j=1}^{N} \frac{1}{x - a_j}\Big)^2 < 0.$$

q.e.d.

One of the most important properties of functions which have negative Schwarzian derivative is the fact that this property is preserved under composition.

Proposition 11.3. *Suppose $Sf < 0$ and $Sg < 0$. Then $S(f \circ g) < 0$.*

Proof. Using the chain rule, one computes that

$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x)$$

and

$$(f \circ g)'''(x) = f'''(g(x)) \cdot (g'(x))^3 + 3f''(g(x)) \cdot g''(x) \cdot g'(x) + f'(g(x)) \cdot g'''(x).$$

It follows that

$$S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x)$$

so that $S(f \circ g)(x) < 0$.

q.e.d.

Of primary importance for us is the immediate consequence that, if $Sf < 0$, then $Sf^n < 0$ for all $n > 1$. The assumption that $Sf < 0$ has surprising implications for the dynamics of a one-dimensional map. One of the major results of this section is

Theorem 11.4. *Suppose $Sf < 0$ ($Sf(x) = -\infty$ is allowed.) Suppose f has n critical points. Then f has* at most $n + 2$ *attracting periodic orbits.*

Remarks.

1. The quadratic function $F_\mu(x) = \mu x(1-x)$ has one critical point ($x = 1/2$). Hence, for each μ there exists *at most three* attracting periodic orbits. There may, of course, be none, as is the case for $\mu > 2 + \sqrt{5}$. Later we will see that the number of attracting periodic orbits can be reduced to at most one. Since, for large μ, the map F_μ has infinitely many periodic orbits, it is indeed a surprise that at most one may be attracting.

2. This presents a computational dilemma. Suppose F_μ has an attracting periodic cycle of period three. By Sarkovskii's Theorem, F_μ must have periodic points of all other periods, but none of them can be attracting. On a computer, only attracting periodic points are "visible," so this raises the question: where are all of the other periodic points in this case? We will return to this question in §1.13.

3. The proofs below extend to non-hyperbolic periodic points as well. Consequently, the quadratic map F_μ has at most one periodic orbit which is *not* repelling.

To prove Theorem 11.4, we first need several lemmas.

Lemma 11.5. *If $Sf < 0$, then $f'(x)$ cannot have a positive local minimum or a negative local maximum.*

Proof. Suppose x_0 is a critical point of $f'(x)$, i.e., $f''(x_0) = 0$. Since $Sf(x_0) < 0$, we have $f'''(x_0)/f'(x_0) < 0$ so that $f'''(x_0)$ and $f'(x_0)$ have opposite signs.

q.e.d.

It follows that, between any two successive critical points of f', the graph of $f'(x)$ must cross the x-axis. In particular, there must be a critical point *for* f between these two points.

Lemma 11.6. *If $f(x)$ has finitely many critical points, then so does $f^m(x)$.*

Proof. For any $c, f^{-1}(c)$ is a finite set of points, since, between any two preimages of c, there must be at least one critical point of f. It follows easily that $f^{-m}(c) = \{x | f^m(x) = c\}$ is also a finite set.

Now suppose $(f^m)'(x) = 0$. By the Chain Rule, we have

$$(f^m)'(x) = \prod_{i=0}^{m-1} f'(f^i(x)).$$