```
130 NEXT I
140 LET C = X: PRINT C
150 INPUT DELTA
160 FOR I = 1 TO N
170 LET X = XL + I*(XU - XL)/N
180 FOR J = 1 TO 20
190 LET X = FNF(X)
200 IF ABS(X - C) < DELTA THEN put a dot at (x, x_e): draw
    a line to (x, x_e + j(x_u - x_e)/50): GOTO 220
210 NEXT J
220 NEXT I
230 DEF FNF(X) = (X*X*X + 6)/7
```

Here lines 100–140 allow the user to input a value of x near the proposed fixed point c and compute a value for c. Since it is essential to obtain a reasonably accurate value for c, the number of iterations at line 110 may need to be increased. Then at line 150 we input a positive number δ which defines the target interval $I = (c - \delta, c + \delta)$. Finally, for each initial value $x_0 = x$ used in the earlier plotting of $y = f(x)$, it calculates M (= 20) terms of the sequence $x_n = f^n(x)$. At line 200 the term x_n is tested to see whether it lies inside the target interval and, if so, a vertical line segment is drawn, whose height indicates how many iterations were required. Note that $x_e = y_e$ and $x_u = y_u$ as in Program 3.1.

6.3 *Exercises*

(*a*) Use Program 6.2 to plot the intervals of attraction for:
 (i) $f(x) = (3x + 2)/(x + 2)$, $c = 2$, $\delta = 0.01$;
 (ii) $f(x) = 2.5x(1 - x)$, $c = 0.6$, $\delta = 0.01$;
 (iii) $f(x) = (\sqrt{2})^x$, $c = 2$, $\delta = 0.01$.

(*b*) Repeat (*a*) with various different values of δ.

(*c*) Try to use Program 6.2 when c is some indifferent fixed point. Why do you run into difficulties?

Warning For some functions, such as $f(x) = (\sqrt{2})^x$, repeated iteration with certain values of x can lead to numbers which are too large for the computer to handle. In this case you may like to include a line such as

```
195 IF ABS(X) > 50 THEN GOTO 220
```

so that such values of x are excluded from consideration and the program can continue. The particular number 50 used in line 195 will of course depend on the function, the bounds x_e, x_u, etc.

§7 *Conjugate functions*

If you attempted Exercise 3.2, then you may have noticed that certain iteration sequences display remarkably similar behaviour despite being defined by entirely different functions. Consider, for example, the sequences:

$$x_{n+1} = x_n(1 - x_n), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $x_0 = \frac{1}{2}$, and

$$x_{n+1} = x_n^2 + \tfrac{1}{4}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $x_0 = 0$. See Fig. 19. It appears that Fig. 19(*b*) can be obtained from Fig. 19(*a*) by rotating about the origin through 180° and then translating (0, 0) to $(\frac{1}{2}, \frac{1}{2})$ (or, what is equivalent, rotating about the point $(\frac{1}{4}, \frac{1}{4})$ through 180°). In fact this geometric effect is achieved by the change of variable:

$$x = -u + \tfrac{1}{2},$$

and if we write $x_n = -u_n + \frac{1}{2}$, $x_{n+1} = -u_{n+1} + \frac{1}{2}$, then Equation (1) becomes

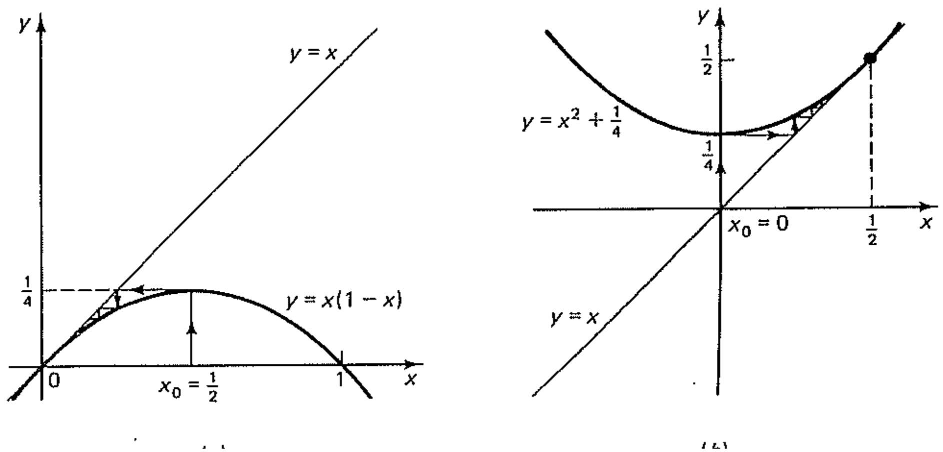
$$\begin{aligned} -u_{n+1} + \tfrac{1}{2} &= (-u_n + \tfrac{1}{2})(\tfrac{1}{2} + u_n) \\ &= \tfrac{1}{4} - u_n^2, \end{aligned}$$

i.e.

$$u_{n+1} = u_n^2 + \tfrac{1}{4},$$

which is precisely Equation (2). Notice also that $u_0 = -x_0 + \frac{1}{2} = 0$, so that the initial terms do correspond. It follows that any known information about sequence (1) can be applied to sequence (2), by using this change of variable.

Fig. 19



We can express this change of variable in function notation as follows. First write Equation (1) in the form

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $f(x) = x(1-x)$ and $x_0 = \frac{1}{2}$. Then put $x = \phi(u)$, where

$$\phi(u) = -u + \tfrac{1}{2}. \tag{4}$$

Now Equation (3) can be written as

$$\phi(u_{n+1}) = f(\phi(u_n)), \quad n = 0, 1, 2, \ldots,$$

i.e.

$$\begin{aligned} u_{n+1} &= \phi^{-1}(f(\phi(u_n))) \\ &= g(u_n), \end{aligned}$$

where

$$g = \phi^{-1} \circ f \circ \phi. \tag{5}$$

Here ϕ^{-1} is the inverse function of ϕ. Schematically we have Fig. 20, in which $x_n = \phi(u_n)$, $x_{n+1} = \phi(u_{n+1})$. In our example

$$x = -u + \tfrac{1}{2} \quad \text{implies} \quad u = -x + \tfrac{1}{2},$$

so that the inverse of the function ϕ in (4) is

$$\phi^{-1}(x) = -x + \tfrac{1}{2}.$$

Hence, by Equation (5)

$$\begin{aligned} g(u) &= -f(\phi(u)) + \tfrac{1}{2} \\ &= -\phi(u)(1 - \phi(u)) + \tfrac{1}{2} \\ &= -(-u + \tfrac{1}{2})(\tfrac{1}{2} + u) + \tfrac{1}{2} \\ &= u^2 + \tfrac{1}{4}, \end{aligned}$$

as expected.

Whenever two functions f and g are related by a formula

$$g = \phi^{-1} \circ f \circ \phi,$$

in which ϕ is a 1–1 correspondence with both ϕ and ϕ^{-1} continuous, we say that f and g are *conjugate* to each other. The two functions f and g can then be considered equivalent, as far as iteration is concerned. Whatever behaviour is exhibited by a sequence

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$

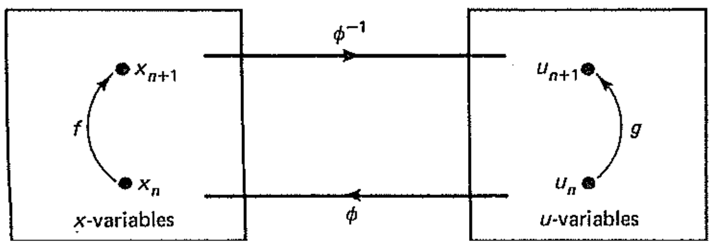
will also be exhibited by

$$u_{n+1} = g(u_n), \quad n = 0, 1, 2, \ldots,$$

where $u_0 = \phi^{-1}(x_0)$, and we say that the sequence u_n is *conjugate* to the sequence x_n.

In the following sections we shall see several examples where the consideration of a conjugate sequence greatly simplifies the understanding of a particular iteration sequence. Sometimes it is even possible to find a formula for x_n by finding a suitable conjugate sequence which can itself be solved explicitly. (See also Chapter 2, Section 6.10.)

Fig. 20



7.1 *Exercises*

(*a*) Find an explicit solution for the iteration sequence

$$x_{n+1} = x_n^2, \quad n = 0, 1, 2, \ldots,$$

with initial term x_0.

(*b*) Consider the sequence

$$x_{n+1} = x_n^2 + 2x_n, \quad n = 0, 1, 2, \ldots,$$

with initial term x_0. By using the change of variables $x = u - 1$ show that

$$x_n = (x_0 - 1)^{2^n} + 1, \quad n = 0, 1, 2, \ldots.$$

7.2 *Exercise*

Suppose that f, g are conjugate functions with

$$g = \phi^{-1} \circ f \circ \phi.$$

Show that if c is a fixed point of f, then $\phi^{-1}(c)$ is a fixed point of g.

§8 *Linear and Möbius sequences*

In §2 and §3 you investigated the behaviour of various linear sequences of the form

$$x_{n+1} = ax_n + b, \quad n = 0, 1, 2, \ldots, \tag{1}$$

and you should have found that the behaviour of such sequences usually depends on the value of a or (if $a = 1$) on the value of b, rather than on the initial term x_0. For example, if $|a| < 1$, then the sequence (1) is always convergent. This can easily be explained graphically since, if $|a| < 1$ then:

(*a*) $y = ax + b$, crosses $y = x$ at a unique fixed point c of $f(x) = ax + b$, given by solving $ax + b = x$, which gives $c = -b/(a-1)$;

(*b*) the derivative of f at this fixed point c is equal to a, and so the fixed point is attracting. See Fig. 21.

Actually we can do rather better than this. The method of conjugate functions can be used in this case to find an explicit formula for x_n. We make the change of variables

$$x = u - b/(a-1), \quad a \neq 1,$$

which takes the fixed point $c = -b/(a-1)$ of f to $u = 0$. Substitution in Equation (1) gives

$$u_{n+1} - \frac{b}{a-1} = a\left(u_n - \frac{b}{a-1}\right) + b, \quad n = 0, 1, 2, \ldots$$

which reduces after cancellation to

$$u_{n+1} = au_n, \quad n = 0, 1, 2, \ldots, \qquad (2)$$

The explicit solution to (2) is

$$u_n = a^n u_0, \quad n = 0, 1, 2, \ldots,$$

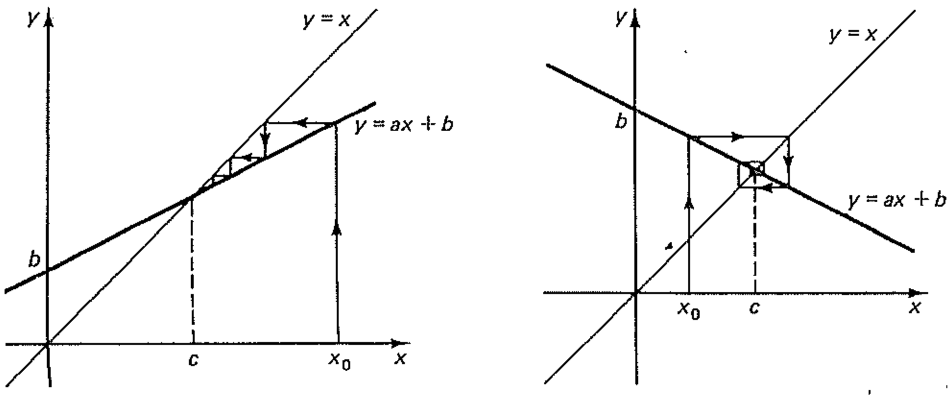
which leads to

$$x_n + \frac{b}{a-1} = a^n\left(x_0 + \frac{b}{a-1}\right),$$

i.e.

$$x_n = a^n x_0 + b\left(\frac{a^n - 1}{a-1}\right). \qquad (3)$$

Fig. 21



(a) $0 < a < 1$ (b) $-1 < a < 0$

This explicit formula for x_n means that we need no help from th micro in order fully to explain the behaviour of sequences of th form (1). For example, it is clear that if $|a| < 1$, then $a^n \to 0$ a $n \to \infty$, and so

$$x_n \to -b/(a-1) \quad \text{as } n \to \infty.$$

8.1 *Exercises*

(*a*) Use Formula (3) to determine the behaviour of x_n if $|a| > 1$.

(*b*) If $a = 1$, then Formula (3) is not valid since $a - 1 = 0$. Deter mine an explicit formula for x_n in this case and deduce tha $x_n \to \pm\infty$ as $n \to \infty$, if $b \neq 0$. Explain this behaviou graphically.

Before moving on to Möbius sequences, which are rather mor complicated, we cannot resist including an application of Formul (3) to the solution of a variant of a puzzle which we met in Chapter 1

8.2 *Exercise: Monkey puzzle*

(Compare Chapter 1, Exercise 2.2.) Five men and a monkey gathe coconuts all day and then fall asleep. During the night each mar wakes in turn and, after giving one coconut to the monkey, he removes and hides one fifth of the pile of coconuts for himself. In the morning the men divide the remainder equally among themselve leaving exactly one coconut, which again goes to the monkey. How many coconuts did they collect? (Hint: each man's nocturnal activity can be described mathematically as an application of the function

$$f(x) = \tfrac{4}{5}(x - 1),$$

to the number x of coconuts which they found.)

Remark For an account of the generalized problem with n men and m monkeys, each of which receives p coconuts, see Melzak (1973), p. 51.

In §2 and §3 you should also have investigated various Möbius sequences of the form

$$x_{n+1} = (ax_n + b)/(cx_n + d), \quad n = 0, 1, 2, \ldots, \qquad (4)$$

where a, b, c, d are real numbers. Such a sequence reduces to a linear sequence if $c = 0$ and to a constant sequence if $ad - bc = 0$. Thus we assume that

$$c \neq 0 \quad \text{and} \quad ad - bc \neq 0.$$

You should have discovered that the behaviour of such sequences is again usually independent of the initial term x_0. For example, if

$$x_{n+1} = \frac{3x_n + 2}{x_n + 2}, \quad n = 0, 1, 2, \ldots,$$

then $x_n \to 2$ for almost all initial terms x_0. See Fig. 22. The only exceptions are $x_0 = -2$ where $f(x) = (3x+2)/(x+2)$ is not defined, and $x = -1$, which is a fixed point of f. How do we explain this behaviour?

You should also have found, in Exercise 2.3, examples of Möbius sequences which are divergent for every choice of initial term x_0. Can this be explained?

Once again it turns out that there is an explicit formula for x_n, which can be used to confirm the observations made earlier. We invite you to derive this somewhat more complicated formula in the following (extended) exercise.

8.3 *Exercise*

(*a*) Determine the fixed points of

$$f(x) = (ax+b)/(cx+d), \quad c \neq 0, \; ad - bc \neq 0.$$

Show that if

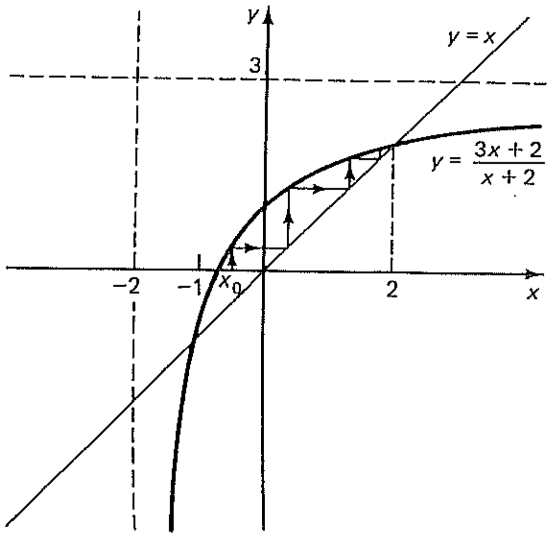
$$(a-d)^2 + 4bc > 0,$$

then f has two distinct fixed points, α, β say.

(*b*) By considering the product

$$(c\alpha + d)(c\beta + d)$$

Fig. 22



show that $c\alpha + d \neq 0$ and $c\beta + d \neq 0$.

(*c*) Use the facts that $\alpha = f(\alpha)$ and $\beta = f(\beta)$ to verify the equatio

$$\frac{x_{n+1} - \alpha}{x_{n+1} - \beta} = \left(\frac{c\beta + d}{c\alpha + d}\right)\left(\frac{x_n - \alpha}{x_n - \beta}\right), \quad \text{if } x_n \neq \beta. \qquad (5$$

(*d*) Deduce from (*c*) that

$$\frac{x_n - \alpha}{x_n - \beta} = \left(\frac{c\beta + d}{c\alpha + d}\right)^n \left(\frac{x_0 - \alpha}{x_0 - \beta}\right), \quad n = 0, 1, 2, \ldots.$$

(*e*) Show that

$$f'(\alpha) = \frac{c\beta + d}{c\alpha + d} \quad \text{and} \quad f'(\beta) = \frac{c\alpha + d}{c\beta + d},$$

and deduce that if $|f'(\alpha)| < 1$, then

$$x_n \to \alpha \quad \text{as } n \to \infty,$$

for all real numbers x_0, apart from β and $-d/c$.

(*f*) Show that if

$$(a-d)^2 + 4bc = 0,$$

then

$$(a+d)^2 = 4(ad - bc)$$

and deduce that $f'(\alpha) = 1$. Show further that the change c variables

$$u = 1/(x - \alpha)$$

transforms the equation of part (*c*) into

$$u_{n+1} = u_n + 2c/(a+d).$$

Deduce that $u_n \to \pm\infty$ as $n \to \infty$, and hence that $x_n \to \alpha$ a $n \to \infty$, in this case.

(*g*) If

$$(a-d)^2 + 4bc < 0,$$

then f has no *real* fixed points and so x_n cannot be convergent However, if α, β denote the *complex* fixed points of f, the Equation (5) is valid. Prove that

$$\left|\frac{c\beta + d}{c\alpha + d}\right| = 1$$

in this case, and hence describe the different possible ways i which x_n is divergent.