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We can express this change of variable in function notation as follows. First write Equation (1) in the form

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $f(x) = x(1-x)$ and $x_0 = \frac{1}{2}$. Then put $x = \phi(u)$, where

$$\phi(u) = -u + \tfrac{1}{2}. \tag{4}$$

Now Equation (3) can be written as

$$\phi(u_{n+1}) = f(\phi(u_n)), \quad n = 0, 1, 2, \ldots,$$

i.e.

$$\begin{aligned} u_{n+1} &= \phi^{-1}(f(\phi(u_n))) \\ &= g(u_n), \end{aligned}$$

where

$$g = \phi^{-1} \circ f \circ \phi. \tag{5}$$

Here ϕ^{-1} is the inverse function of ϕ. Schematically we have Fig. 20, in which $x_n = \phi(u_n)$, $x_{n+1} = \phi(u_{n+1})$. In our example

$$x = -u + \tfrac{1}{2} \quad \text{implies} \quad u = -x + \tfrac{1}{2},$$

so that the inverse of the function ϕ in (4) is

$$\phi^{-1}(x) = -x + \tfrac{1}{2}..$$

Hence, by Equation (5)

$$\begin{aligned} g(u) &= -f(\phi(u)) + \tfrac{1}{2} \\ &= -\phi(u)(1 - \phi(u)) + \tfrac{1}{2} \\ &= -(-u + \tfrac{1}{2})(\tfrac{1}{2} + u) + \tfrac{1}{2} \\ &= u^2 + \tfrac{1}{4}, \end{aligned}$$

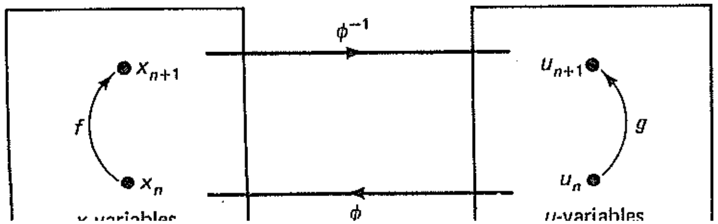
as expected.

Whenever two functions f and g are related by a formula

$$g = \phi^{-1} \circ f \circ \phi.$$

in which ϕ is a 1–1 correspondence with both ϕ and ϕ^{-1} continuous, we say that f and g are *conjugate* to each other. The two functions f

Fig. 20



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and g can then be considered equivalent, as far as iteration is concerned. Whatever behaviour is exhibited by a sequence

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$

will also be exhibited by

$$u_{n+1} = g(u_n), \quad n = 0, 1, 2, \ldots,$$

where $u_0 = \phi^{-1}(x_0)$, and we say that the sequence u_n is *conjugate* to the sequence x_n.

In the following sections we shall see several examples where the consideration of a conjugate sequence greatly simplifies the understanding of a particular iteration sequence. Sometimes it is even possible to find a formula for x_n by finding a suitable conjugate sequence which can itself be solved explicitly. (See also Chapter 2, Section 6.10.)

7.1 *Exercises*

(*a*) Find an explicit solution for the iteration sequence

$$x_{n+1} = x_n^2, \quad n = 0, 1, 2, \ldots,$$

with initial term x_0.

(*b*) Consider the sequence

$$x_{n+1} = x_n^2 + 2x_n, \quad n = 0, 1, 2, \ldots,$$

with initial term x_0. By using the change of variables $x = u - 1$ show that

$$x_n = (x_0 - 1)^{2^n} + 1, \quad n = 0, 1, 2, \ldots.$$

7.2 *Exercise*

Suppose that f, g are conjugate functions with

$$g = \phi^{-1} \circ f \circ \phi.$$

Show that if c is a fixed point of f, then $\phi^{-1}(c)$ is a fixed point of g.

§8 *Linear and Möbius sequences*

In §2 and §3 you investigated the behaviour of various linear sequences of the form

$$x_{n+1} = ax_n + b, \quad n = 0, 1, 2, \ldots, \tag{1}$$

and you should have found that the behaviour of such sequences usually depends on the value of a or (if $a = 1$) on the value of b, rather than on the initial term x_0. For example, if $|a| < 1$, then the

Use the results of Exercise 8.3 to give a complete description of the behaviour of the Möbius sequences which were investigated in Exercise 2.3. Hence confirm that the continued fraction [1, 2, 2, . . .] converges to $\sqrt{2}$ (see §1).

§9 *Quadratic sequences*

The study of linear sequences and Möbius sequences in the previous section was rather straightforward because of the availability of explicit solutions. For quadratic sequences of the general form

$$x_{n+1} = ax_n^2 + bx_n + c, \quad n = 0, 1, 2, \ldots, \tag{1}$$

no explicit solution is available. There are some quadratic sequences for which an explicit solution can be found; for example the sequence

$$x_{n+1} = x_n^2, \quad n = 0, 1, 2, \ldots,$$

has the explicit solution

$$x_n = x_0^{2^n}, \quad n = 0, 1, 2, \ldots,$$

but usually other methods are required to determine the behaviour of a quadratic sequence.

In this section we shall focus attention on quadratic sequences of the form

$$x_{n+1} = \lambda x_n(1 - x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

which arise in population dynamics, as we described in §1. See Fig. 23. Such sequences may seem much more special than (1), since there is only one parameter, namely λ, but in fact they are not all that special. Whenever the function $f(x) = ax^2 + bx + c$, which defines sequence (1), has a real fixed point, the sequence (1) is actually conjugate to a sequence of the form (2). This is the content of the following exercise.

9.1 *Exercise*

Suppose that $f(x) = ax^2 + bx + c$, $a \neq 0$, has the fixed point α and that $f'(\alpha) \neq 0$. Prove that the sequence (1) is conjugate to sequence (2) using the change of variable

$$x = -\left(\frac{2\alpha a + b}{a}\right)u + \alpha;$$

the corresponding value of λ is $\lambda = 2\alpha a + b = f'(\alpha)$.

9.2 *A systematic investigation*

In Exercise 2.4 you calculated the sequence

$$x_{n+1} = \lambda x_n(1 - x_n), \quad n = 0, 1, 2, \ldots,$$

various initial values x_0. This range of values of λ is of particula interest because for $0 \leqslant \lambda \leqslant 4$ the function f_λ, defined by

$$f_\lambda(x) = \lambda x(1 - x)$$

maps the interval [0, 1] into itself. Indeed you can readily check tha f_λ takes its maximum value $\lambda/4$ at the point $x = \frac{1}{2}$ (Fig. 23).

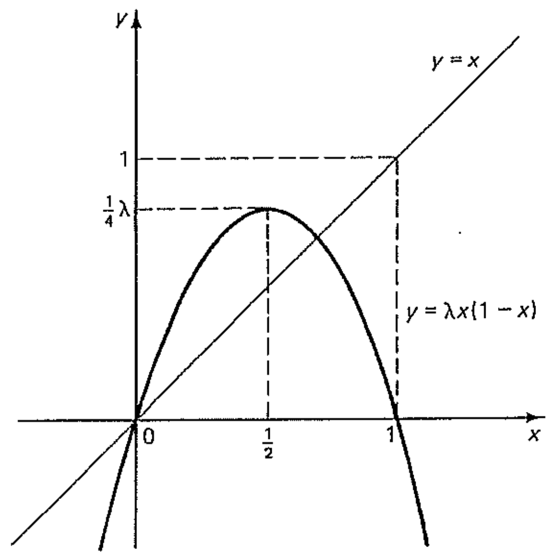
You should have discovered that the behaviour of the sequence x varies very greatly depending on the value of λ, but for each fixed the behaviour is more or less independent of the initial term x_0. Fo example there are many values of λ for which the sequence i convergent and others for which it appears to converge to a p-cycl ($p > 1$). The time has now come for a systematic investigation c such sequences.

The program below takes a large number of values of λ between and 4 and plots each of the corresponding sequences x_n (startin from $x_0 = \frac{1}{2}$) vertically above a horizontal λ-axis. In order to detec the eventual behaviour of x_n (where possible), only the terms x_{50}, x_5 . . ., x_{100} are plotted. Thus if x_n is convergent, then a single poir should be plotted and, more generally, if x_n converges to a p-cycl then p points should be plotted.

9.3 *Program: Lambda plot*

```
1Ø LET N=Ø.25*VMAX
2Ø FOR I=1 TO N
```

Fig. 23



```
4Ø LET X = Ø.5
5Ø FOR J = 1 TO 1ØØ
6Ø LET X = L*X*(1 - X)
7Ø IF J > 49 THEN put a dot at screen point (4*I, VMAX*X)
8Ø NEXT J
9Ø NEXT I
```

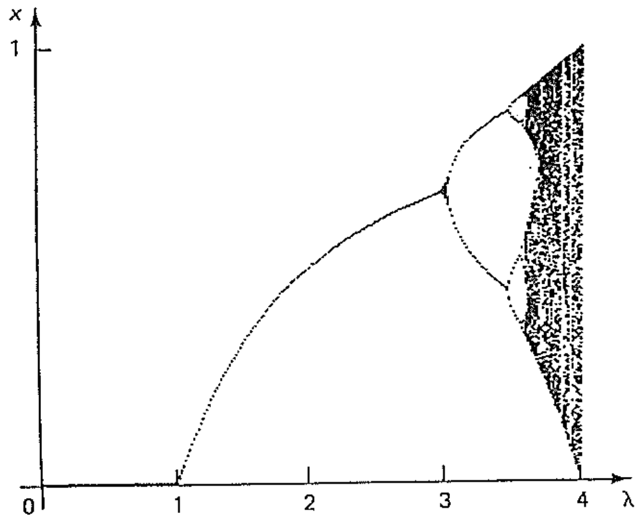
Here VMAX denotes the maximum vertical screen coordinate and we have taken only $n = \frac{1}{4}v_{max}$ values of λ (denoted by the variable l) because the program is quite slow. After trying the program out, you may like to increase n.

Note: If your micro has the graphics origin at the top left of the screen, then, to get the picture the usual way up, you will need to use (4*I, VMAX*(1 − X)) in line 7Ø. Similar remarks hold for Programs 9.21 and 10.1.

The remarkable picture which emerges is shown in Fig. 24. We have added axes in order to measure where the different types of behaviour occur. The picture reveals the following:

for $0 \leqslant \lambda \leqslant 1$, the sequence x_n is convergent with limit 0;
for $1 < \lambda \leqslant 3$, the sequence x_n is convergent with non-zero limit;
for $3 < \lambda \leqslant 3.45$ (approximately) the sequence converges to a 2-cycle;
as λ increases beyond 3.45, the sequence x_n appears first to converge to a 4-cycle and then to behave in a rather chaotic manner.

Fig. 24



range $3 \leqslant \lambda \leqslant 4$.

9.4 *Exercise*

(*a*) Modify Program 9.3 so that λ varies from 3 to 4, by altering line 3Ø:

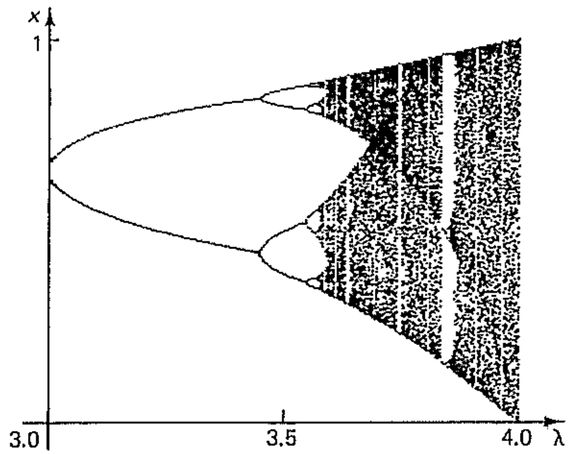
3Ø L = 3 + I/N.

(*b*) Make a similar modification to look in detail at $3.5 \leqslant \lambda \leqslant 4$.

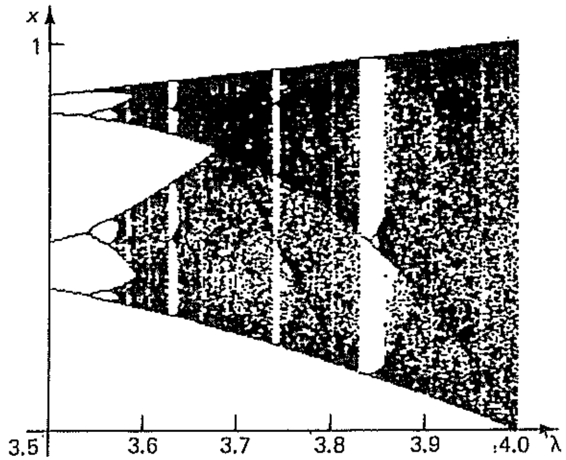
Fig. 25 is even more remarkable:

for $3.45 \leqslant \lambda \leqslant 3.55$ (approximately) the sequence x_n converges to a 4-cycle;

Fig. 25



(*a*)



(*b*)

ges to an 8-cycle, and thereafter there appears to be a 16-cycle;
for λ close to 3.63 (approximately) the sequence x_n converges to a 6-cycle;
for λ close to 3.74 (approximately) the sequence x_n converges to a 5-cycle;
for λ close to 3.83 (approximately) the sequence x_n converges to a 3-cycle;
for other values of λ it appears that x_n is chaotic.

9.5 *Exercise*

Modify Program 9.3 so that λ lies in various narrow intervals between 3.5 and 4. Can you find another interval of values of λ in which x_n converges to a 4-cycle?

The aim of this section is to attempt to elucidate some of the remarkable behaviour contained in the above pictures. We should warn you, however, that this behaviour is not completely understood and is still the subject of much current research. In case you are wondering why it is worth devoting great effort to understanding such a specialised type of sequence we recommend that you try the next exercise, which deals with the iteration of an entirely different family of functions, which have roughly the same shape of graph as that of $f_\lambda(x)$ over $0 \leqslant x \leqslant 1$.

9.6 *Exercise*

Use Program 9.3, with an appropriate modification, to investigate the behaviour of the sequences

$$x_{n+1} = \lambda \sin(\pi x_n), \quad n = 0, 1, 2, \ldots, 0 \leqslant \lambda \leqslant 1,$$

with $x_0 = \frac{1}{2}$. (Here πx_n is in radians.)

9.7 *The range* $0 \leqslant \lambda \leqslant 3$

In this section we verify the observed behaviour of

$$x_{n+1} = \lambda x_n(1 - x_n), \quad n = 0, 1, 2, \ldots,$$

with $x_0 = \frac{1}{2}$, for λ in the range $0 \leqslant \lambda \leqslant 3$; i.e. we shall prove that:

for $0 \leqslant \lambda \leqslant 1$, the sequence x_n converges to 0;
for $1 < \lambda \leqslant 3$, the sequence x_n tends to a non-zero limit.

First we note that the fixed points of $f_\lambda(x) = \lambda x(1 - x)$ are the solutions of

$$\lambda x(1 - x) = x;$$

$$x = 0 \quad \text{and} \quad x = 1 - 1/\lambda = c_\lambda,$$

say. Thus for $0 < \lambda \leqslant 1$, the function f_λ has no fixed points between and 1. Since $f_\lambda(0) = f_\lambda(1) = 0$, we deduce that the graph $y = f_\lambda(x)$ li below $y = x$, for $0 < x < 1$, i.e.

$$0 < f_\lambda(x) < x, \quad \text{for } 0 < x < 1, 0 < \lambda \leqslant 1.$$

It follows by graphical iteration that if we take *any* initial term with $0 < x_0 < 1$, then the sequence x_n converges to 0. See Fig. 26.

Next we consider the case $1 < \lambda \leqslant 2$. For this range, the fixed poi $c_\lambda = 1 - 1/\lambda$ of f_λ satisfies $0 < c_\lambda \leqslant \frac{1}{2}$ and the graph of f_λ looks as i

Fig. 26

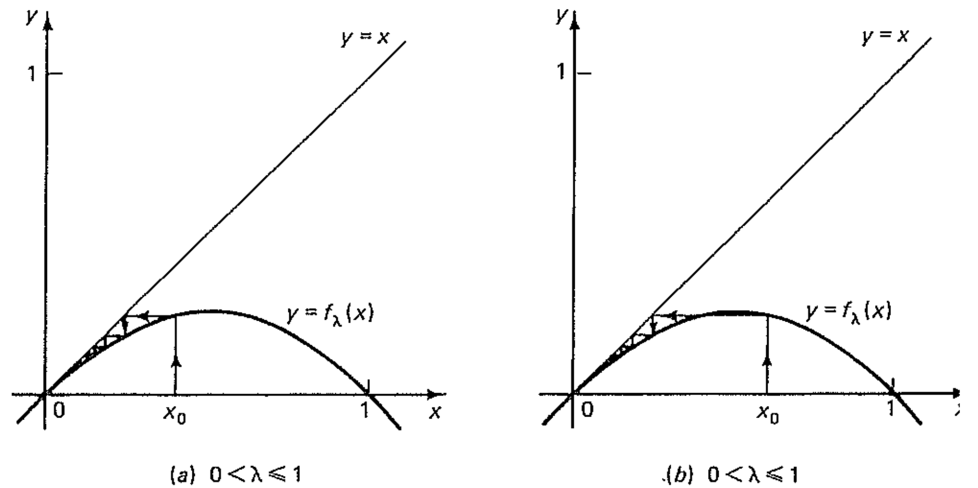


Fig. 27

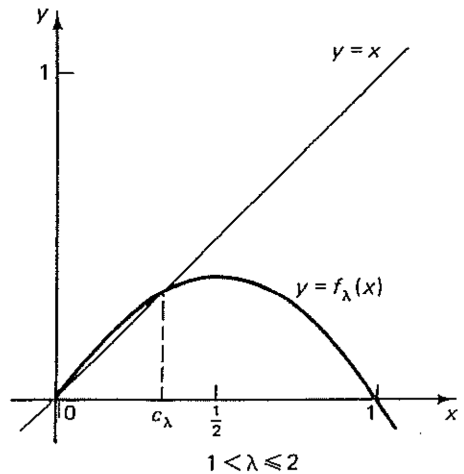


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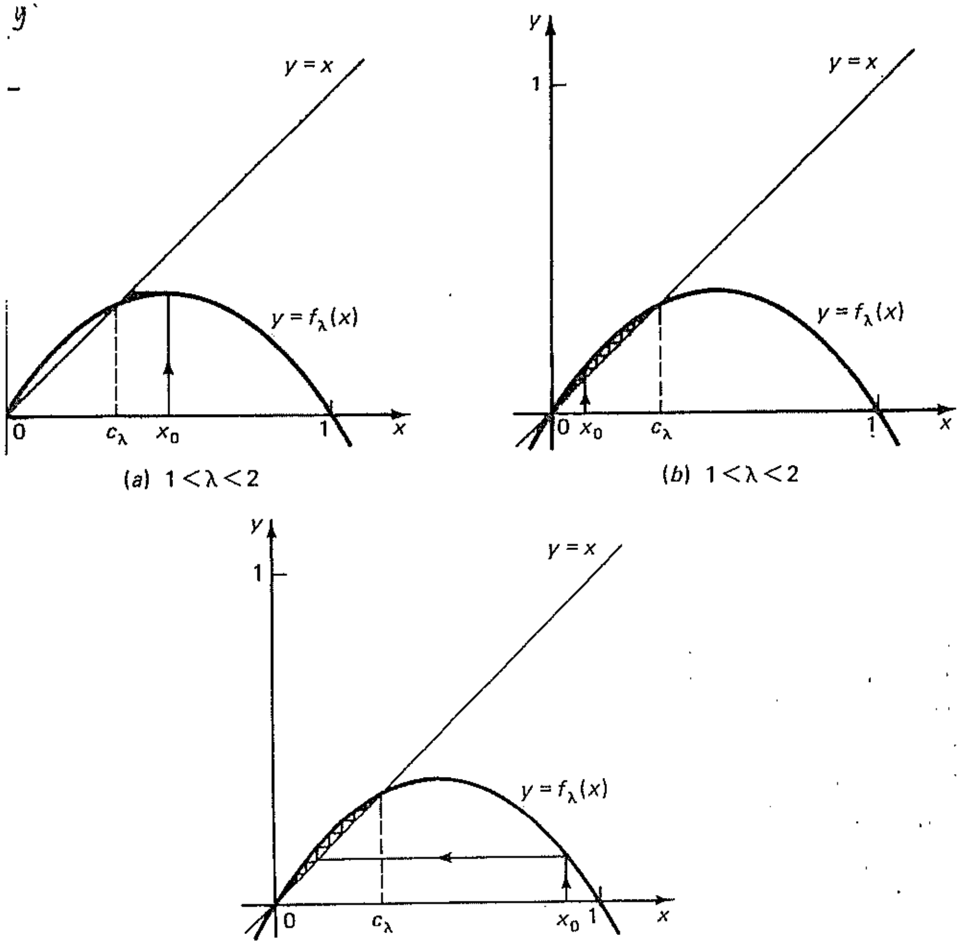
Fig. 27. Once again it is clear in this case that if $0 < x_0 < 1$, then x_n tends to $c_\lambda = 1 - 1/\lambda$. See Fig. 28.

Incidentally, this convergence shows that in Fig. 24 the part of the plot above the λ-interval [1, 2] has the equation $x = 1 - 1/\lambda$. Since the part of the plot above the λ-interval [2, 3] seems also to lie on the same curve, we expect that the sequence x_n (with $x_0 = \frac{1}{2}$) converges to c_λ for $2 < \lambda \leqslant 3$ also.

In the range $2 < \lambda \leqslant 3$ the fixed point $c_\lambda = 1 - 1/\lambda$ satisfies $\frac{1}{2} < c_\lambda \leqslant \frac{2}{3}$ and so the graph $y = f_\lambda(x)$ is decreasing as it passes through c_λ. See Fig. 29. This suggests that the sequence $x_{n+1} = f_\lambda(x_n)$, $n = 0, 1, 2, \ldots, x_0 = \frac{1}{2}$, will yield a cobweb around c_λ, in other words

Fig. 28



(a) $1 < \lambda < 2$

(b) $1 < \lambda < 2$

(c) $1 < \lambda < 2$

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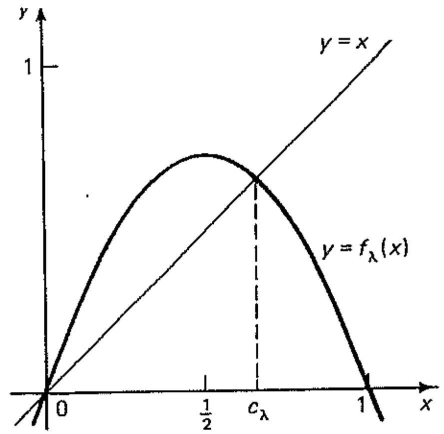
Fig. 29



$2 < \lambda \leqslant 3$

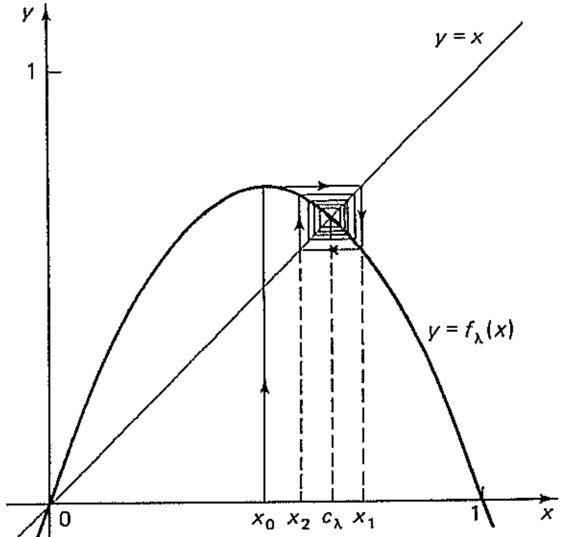
$$x_0 < x_2 < x_4 < \ldots < c_\lambda < \ldots < x_5 < x_3 < x_1.$$

See Fig. 30. Although the sequence x_n does appear to converge to c_λ it is not clear how to *prove* this. The next few sections (up to Exercise 9.10) are devoted to the proof.

First note that

$$\begin{aligned} f'_\lambda(x)|_{c_\lambda} &= \lambda(1-2x)|_{c_\lambda} \\ &= \lambda(1 - 2(1 - 1/\lambda)) \\ &= 2 - \lambda, \end{aligned}$$

Fig. 30



$2 < \lambda \leqslant 3$

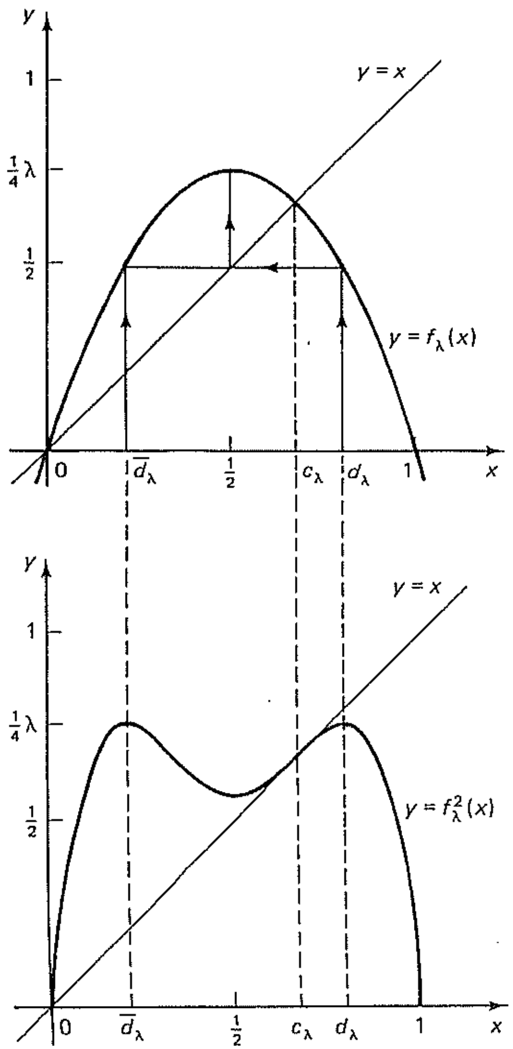
so that

$$|f'_\lambda(c_\lambda)| = |2 - \lambda|.$$

Hence c_λ is not a repelling fixed point for $2 < \lambda \leq 3$. It is attracting if $2 < \lambda < 3$ and indifferent if $\lambda = 3$. We know from §4 that if c_λ is attracting then x_n will converge to c_λ if it lands *close enough* to c_λ. But it is not immediately clear that this will happen if $x_0 = \frac{1}{2}$. Thus, some extra argument is required here.

The key idea is to look at the function $f^2_\lambda(x) = f_\lambda(f_\lambda(x))$. The graph of this function can be plotted using Program 5.1, with $p = 2$, and for $2 < \lambda \leq 3$ it looks as in Fig. 31. Both $y = f_\lambda(x)$ and $y = f^2_\lambda(x)$ are

Fig. 31



plotted in order to make comparisons. There are several key features of $y = f^2_\lambda(x)$, which we now ask you to verify.

9.8 *Exercises*

Let $f_\lambda(x) = \lambda x(1 - x)$ with $2 < \lambda \leq 3$.

(*a*) Prove that f^2_λ is symmetric about the line $x = \frac{1}{2}$; i.e.

$$f^2_\lambda(\tfrac{1}{2} - x) = f^2_\lambda(\tfrac{1}{2} + x).$$

(*b*) Prove that f^2_λ has a fixed point at c_λ, and that

$$(f^2_\lambda)'(c_\lambda) = (f'_\lambda(c_\lambda))^2.$$

(Hint: Use the chain rule.)

(*c*) Prove that f^2_λ takes its maximum value $\lambda/4$ at d_λ and $1 - d_\lambda$, where $d_\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{(1 - 2/\lambda)}$, so that $f_\lambda(d_\lambda) = \frac{1}{2}$.

(*d*) Prove that $c_\lambda < f_\lambda(\frac{1}{2}) < d_\lambda$ and deduce that

$$f^2_\lambda(\tfrac{1}{2}) > \tfrac{1}{2}.$$

(*e*) Prove that f^2_λ is increasing for $\frac{1}{2} \leq x \leq d_\lambda$.

The final feature of $y = f^2_\lambda(x)$ which we require is that f^2_λ has no fixed points in $(0, 1)$ other than c_λ. Put another way, the graph $y = f^2_\lambda(x)$ crosses $y = x$ only once for $0 < x < 1$. There are various ways to prove this fact, which seems so obvious from Fig. 31. The direct method is to write down the fixed point equation

$$f^2_\lambda(x) = x \tag{3}$$

and show that the only solutions are 0 and c_λ if $2 < \lambda \leq 3$. Although this is a quartic equation in x, all four roots can be found easily because we know that 0 and c_λ are solutions. We ask you to carry out this manipulation in the following exercise.

9.9 *Exercises*

(*a*) Prove that

$$f^2_\lambda(x) - x = (f_\lambda(x) - x)g_\lambda(x),$$

where

$$g_\lambda(x) = \lambda^2 x^2 - (\lambda^2 + \lambda)x + (\lambda + 1).$$

(Hint: Begin by writing

$$f^2_\lambda(x) - x = f^2_\lambda(x) - f_\lambda(x) + f_\lambda(x) - x.)$$

(*b*) Show that the solutions of $g_\lambda(x) = 0$ are

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2λ …

and deduce that f_λ^2 has no fixed points in (0, 1), other than c_λ, if $2 < \lambda \leqslant 3$.

Having established that f_λ^2 has no fixed points in (0, 1), apart from c_λ, it is now clear that

$$x < f_\lambda^2(x) < c_\lambda, \quad \text{for } \tfrac{1}{2} \leqslant x < c_\lambda,$$
$$x > f_\lambda^2(x) > c_\lambda, \quad \text{for } c_\lambda < x \leqslant d_\lambda.$$

See Fig. 32.

It follows that, for $\frac{1}{2} \leqslant x \leqslant d_\lambda$,

$$f_\lambda^{2n}(x) \to c_\lambda \quad \text{as } n \to \infty.$$

In particular the sequence $f_\lambda^{2n}(x_0)$, i.e. $x_0, x_2, x_4, \ldots$, converges to c_λ, as does $f_\lambda^{2n}(x_1)$, i.e. $x_1, x_3, x_5, \ldots$, (since $x_1 = f_\lambda(\frac{1}{2})$ lies in (c_λ, d_λ) by Exercise 9.8(*d*)). Thus we have shown that

$$x_n = f_\lambda^n(x_0), \quad n = 0, 1, 2, \ldots,$$

with $x_0 = \frac{1}{2}$, does indeed converge to $c_\lambda = 1 - 1/\lambda$ for $2 < \lambda \leqslant 3$.

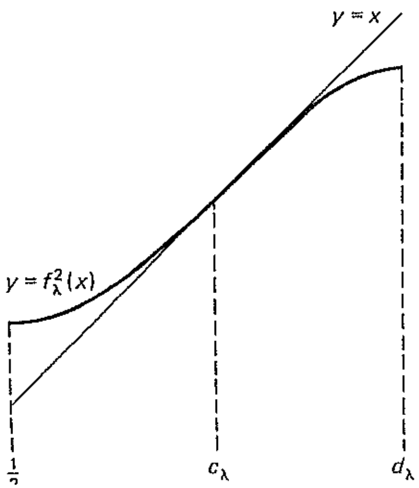
It is natural to ask now whether $f_\lambda^n(x) \to c_\lambda$ for all initial terms x in (0, 1) (if $x = 0$ or 1 then, of course, $f_\lambda^n(x) = 0$, for $n = 1, 2, \ldots$). We ask you to answer this question in the following exercise.

9.10 *Exercise*

Show that if $2 < \lambda \leqslant 3$ and $0 < x < 1$, then

$$f_\lambda^n(x) \to c_\lambda \quad \text{as } n \to \infty.$$

Fig. 32



(Hint: Show that $f_\lambda^n(x)$ lies in $[\frac{1}{2}, a_\lambda]$ for some n depending on x.)

9.11 *What happens if $\lambda > 3$?*

Next we look in detail at what happens to the sequence $x_n = f_\lambda^n(\frac{1}{2})$ as λ increases beyond 3. In our earlier experiments we saw that for $3 < \lambda \leqslant 3.45$ (approx.) x_n converges to a 2-cycle. Why does this happen, and what is the significance of the number 3.45?

First let us see what the sequence x_n looks like (using Program 3.1) for a value of λ slightly greater than 3. See Fig. 33.

Once again we seem to have

$$x_0 < x_2 < x_4 < \ldots < c_\lambda < \ldots < x_5 < x_3 < x_1,$$

but this time the cobweb does not seem to close in on the fixed point c_λ. Let us see if the graph $y = f_\lambda^2(x)$ is any help (Program 5.1, with $p = 2$). Fig. 34 makes it much clearer what is happening. The function $f_\lambda^2(x)$ has acquired a new pair of fixed points, here labelled a_λ and b_λ. Indeed you have already found these two fixed points in Exercise 9.9(*b*), where you solved

$$f_\lambda^2(x) = x$$

to find the solutions 0, c_λ and $(1/2\lambda)\ (\lambda + 1 \pm \sqrt{((\lambda + 1)(\lambda - 3))})$. The last two fixed points are real if $\lambda > 3$ and so we must have

$$a_\lambda = \frac{1}{2\lambda}[\lambda + 1 - \sqrt{((\lambda + 1)(\lambda - 3))}]$$

Fig. 33