All questions are similar to homework problems.

MATH191 Solutions September 2008
Section A

1. To find the inverse function,

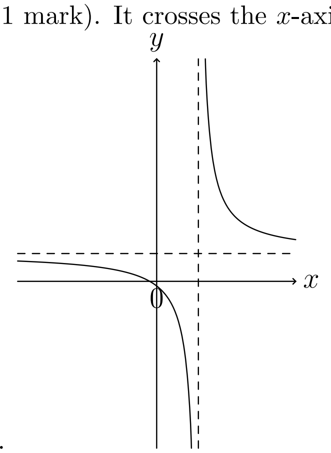
$$y = \frac{2x+1}{x-3} \Leftrightarrow y(x-3) = 2x+1 \Leftrightarrow yx - 3y = 2x+1$$

$$\Leftrightarrow x(y-2) = 3y+1 \Leftrightarrow x = \frac{3y+1}{y-2}.$$

(1 mark)
So the inverse function is given by

$$f^{-1}(y) = \frac{3y+1}{y-2} \quad \text{or} \quad f^{-1}(x) = \frac{3x+1}{x-2}.$$

(1 mark)
The graph of f is shown below (1 mark). It crosses the x-axis at $x = -1/2$



and the y-axis at $y = -\frac{1}{3}$. (1 mark).
[$1+1+1+1 = 4$ marks]

2. We have $f(0) = 0$, $f'(x) = -2(1-2x)^{-1}$ and $f''(x) = -4(1-2x)^{-2}$, so $f'(0) = -2$, and $f''(0) = -4$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).
Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = -2x - (4/2)x^2 + \cdots = 0 - 2x - 2x^2 + \cdots$$

(1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = -2x - 2x^2$).
[$3+1+1 = 5$ marks]

3.

a) $r = \sqrt{1+3} = 2$ (1 mark). $\theta = \frac{2\pi}{3}$ (2 marks).

b) $x = 3\cos(5\pi/3) = \frac{3}{2}$. $y = 3\sin(5\pi/3) = -3\frac{\sqrt{3}}{2}$. (1 mark each)

Subtract one mark for each answer not given exactly.
[$3 + 2 = 5$ marks]

4.

$$\begin{aligned}\int_0^1 ((1+x)^{-1/2} + (1+x)^{-2})dx &= \Big[2(1+x)^{1/2} - (1+x)^{-1}\Big]_0^1 \qquad \text{(3 marks)}\\ &= 2\sqrt{2} - 2^{-1} - 2 + 1) \\ &= 2\sqrt{2} - \frac{3}{2} \qquad \text{(2 marks)}\end{aligned}$$

[$3 + 2 = 5$ marks]

5. Differentiating the equation with respect to x gives

$$3x^2 + y^2 + 2xy\frac{dy}{dx} + \frac{dy}{dx} = 0 \quad \text{(2 marks)}.$$

Hence

$$\frac{dy}{dx} = \frac{-3x^2 - y^2}{1 + 2xy} \quad \text{(2 marks)}.$$

Thus $\frac{dy}{dx}$ is equal to $-\frac{4}{-1} = 4$ when $(x, y) = (-1, 1)$. (2 marks).
The equation of the tangent at this point is therefore

$$y - 1 = -4(x + 1) \qquad \text{or } y = 4x + 5 \qquad \text{(2 marks)}.$$

[$2 + 2 + 2 + 2 = 8$ marks]

6.

a) By the chain rule,

$$\frac{d}{dx}(\cos(x^3)) = 3x^2\sin(x^3). \qquad \text{(2 marks)}.$$

b) By the quotient rule,

$$\frac{d}{dx}\left(\frac{\ln x}{x^2 - 2}\right) = \frac{(x^2 - 2)x^{-1} - 2x\ln x}{(x^2 - 2)^2} = \frac{x^2 - 2 - 2x^2\ln x}{x(x^2 - 2)^2} \qquad \text{(2 marks)}.$$

c) By the chain rule,

$$\frac{d}{dx}\left((x^2 + 2x + 2)^{1/2}\right) = \frac{2}{2}(x+1)(x^2+2x+2)^{-1/2} = \frac{x + 1}{\sqrt{x^2 + 2x + 2}} \qquad \text{(2 marks)}.$$

[$2+2+2=6$ marks]

7. For $f(x)=xe^{-x}$,

$$f'(x)=-xe^{-x}+e^{-x}=e^{-x}(1-x)$$

Stationary points are given by solutions of $f'(x)=0$, So there is exactly one stationary point, namely $x=1$. (3 marks)

To determine its nature,

$$f''(x)=-e^{-x}(1-x)-e^{-x}=(x-2)e^{-x}.$$

So $f''(1)=-e^{-1}<0$, and 1 is a local maximum. (3 marks)
[$3+3=6$ marks]

8.

$$\begin{aligned}
z_1+z_2 &= 4-3j \qquad \text{(1 mark)}\\
z_1-z_2 &= 2+5j \qquad \text{(1 mark)}\\
z_1z_2 &= (3+j)(1-4j)=3-11j-4j^2=7-11j \qquad \text{(2 marks)}\\
z_1/z_2 &= \frac{(3+j)(1+4j)}{(1-4j)(1+4j)}=\frac{3+13j+4j^2}{17}=\frac{-1+13j}{17} \qquad \text{(2 marks).}
\end{aligned}$$

[$1+1+2+2=6$ marks]

9. $\cos^{-1}(-\sqrt{3}/2)=5\pi/6$ (1 mark)

The general solution of $\cos\theta=\frac{-\sqrt{3}}{2}$ is

$$\theta=\pm 5\pi/6+2n\pi$$

for any $n\in\mathbb{Z}$. (3 marks)
[$1+3=4$ marks]

10.

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= 5\mathbf{i}-3\mathbf{j}-2\mathbf{k} \quad \text{(1 mark)}\\
\mathbf{a}-\mathbf{b} &= -\mathbf{i}+\mathbf{j}+6\mathbf{k} \quad \text{(1 mark)}\\
|\mathbf{a}| &= \sqrt{4+1+4}=3 \qquad \text{(1 mark)}\\
|\mathbf{b}| &= \sqrt{3^2+2^2+4^2}=\sqrt{29} \qquad \text{(1 mark)}\\
\mathbf{a}\cdot\mathbf{b} &= 6+2-8=0 \qquad \text{(1 mark).}
\end{aligned}$$

Hence the angle between **a** and **b** is $\pi/2$ (1 mark).
[$1+1+1+1+1+1=6$ marks]

Section B

11. The Maclaurin series expansion of $\sqrt{1+x}$ is

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{64}x^4 + \cdots + \binom{\frac{1}{2}}{n} x^n + \cdots \qquad \text{(2 marks)}$$

Here

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdots (\frac{3}{2} - n)}{n!}$$

(This definition is not required.)

Hence the other Maclaurin series are:

a) for $\sqrt{1-x}$,

$$1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{64}x^4 + \cdots + (-1)^n \binom{\frac{1}{2}}{n} x^n + \cdots \qquad \text{(2 marks)}$$

b) for $\sqrt{4+x} = 2(1+x/4)^{1/2}$,

$$2 + \frac{1}{2}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 - \frac{5}{8192}x^4 + \cdots + 2 \cdot 4^{-n} \binom{\frac{1}{2}}{n} x^n + \cdots \qquad \text{(3 marks)}$$

c) for $\sqrt{1-x^2}$,

$$1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{64}x^8 + \cdots + (-1)^n \binom{\frac{1}{2}}{n} x^{2n} + \cdots \qquad \text{(3 marks)}$$

d) The Maclaurin series of $g(x) = (1-x^2)^{1/2}$ is

$$g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \cdots \frac{g^n(0)}{n!}x^n + \cdots$$

So comparing coefficients of x^n we see that

$$g^{(n)}(0) = \begin{matrix} 0 & \text{if } n \text{ is odd,} \\ (-1)^n n! \binom{\frac{1}{2}}{n} & \text{if } n \text{ is even.} \end{matrix} \qquad \text{(5 marks)}$$

[$2+2+3+3+5 = 15$ marks]

12.

a) The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case

$$|a_n/a_{n+1}| = \frac{2^n(n+2)^2}{2^{n+1}(n+1)^2},$$

which converges to $\frac{1}{2}$. So the radius of convergence is $\frac{1}{2}$. (4 marks)

At $R = \frac{1}{2}$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is convergent. At $R = -\frac{1}{2}$ the series becomes

$$\sum_{n=1}^{\infty} (-1)^n (n+1)^2.$$

which is also convergent (2 marks)

b) In this case $a_n = 2^{-n}(n+1)^{-1/2}$, so $|a_n/a_{n+1}| = 2\sqrt{(n+2)/(n+1)}$, which tends to 2 as $n \to \infty$. Hence $R = 2$. (4 marks)

At $R = 2$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges. At $R = -2$ the series becomes

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$$

which converges as it is an alternating series and $1/\sqrt{n}$ decreases to 0 as $n \to \infty$ (5 marks)

$[4 + 2 + 4 + 5 = 15$ marks$]$

13. For $f(x) = x^3 + x^2 + 2x - 3$, $f'(x) = 3x^2 + 2x + 2 > 0$ for all x because for this quadratic, $(b^2/4) - ac = 1 - 6 < 0$. So f is increasing on $(-\infty, \infty)$

We have

$$f(0) = -3, \quad f(1) = 1.$$

So there is just one zero of f, which is in $(0, 1)$. (6 marks)

The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 + 2x_n - 3}{3x_n^2 + 2x_n + 2} = \qquad \text{(3 marks)}$$

Hence, if $x_0 = 1$, we have $f(x_0) = 1$ and

$$x_1 = \frac{6}{7}, \quad f(x_1) = 0.078717201.. \qquad \text{(1 mark)}$$

$$x_2 = 0.84384236 \qquad f(x_2) = 0.0000629443.. \qquad \text{(2 mark)}$$

$$x_3 = 0.84373428, \qquad f(x_3) = 0.00000004 \qquad \text{(3 marks)}$$

[$6 + 3 + 1 + 2 + 3 = 15$ marks]

14. For horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x+(1-x)^{-1}) = -\infty,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(1-x)^{-1} = 0.$$

So $y=0$ is a horizontal asymptote (although only at $+\infty$). (2 marks)

For vertical asymptotes: the only possible asymptote is where $1-x=0$, that is, where $x=1$. We have

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} 1-x = 0, \quad \lim_{x\to 1+}(1-x)^{-1} = -\infty$$

So $x=1$ is a vertical asymptote. (1 mark)

For points of continuity: the only possible discontinuities are at 1 and 0. 1 is certainly a discontinuity, because it is a vertical asymptote. 0 is not a discontinuity because

$$f(0-) = \lim_{x\to 0}(x+(1-x)^{-1}) = 1 = \lim_{x\to 0}(1-x) = f(0+) = f(0).$$

(2 marks)

We have

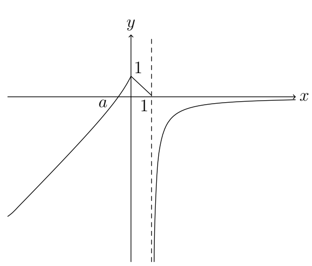
$$f'(x) = \begin{cases} 1+(1-x)^{-2} & \text{if } x\in(-\infty,0) \\ -1 & \text{if } x\in(0,1) \\ (1-x)^{-2} & \text{if } x\in(1,\infty) \end{cases}$$

[2 marks]

The function is not differentiable at 1 because it is not continuous there. The only other point that needs checking is 0. There the function is not differentiable because the left derivative is 2 and the right derivative is -1. (2 marks)

Now $f'(x)>0$ on each of the intervals $(-\infty,0)$, and $(1,\infty)$ and <0 on $(0,1)$. So there are no stationary points. The function f is increasing on each of the intervals $(-\infty,0]$ and $(1,\infty)$ and decreasing on $[0,1]$. (2 marks)

For zeros: since $(1-x)^{-1}\neq 0$ for any x, and $1-x=0 \Leftrightarrow x=1$ for $x\in[0,1]$. The only other possibility is that $x\in(-\infty,0)$ and $x+(1-x)^{-1}=0$, that is, when $x-x^2+1=0$ and $x<0$, that is when $x^2-x-1=0$ and $x<0$. The zeros of x^2-x-1 are $(1\pm\sqrt{5})/2$. Since $1+\sqrt{5}>0$, and $1-\sqrt{5}<0$, the only two zeros of f are 1 and $(1-\sqrt{5})/2$ (2 marks)



The graph of f is as shown, with $a=(1-\sqrt{5})/2$.
[2 marks]
[$2+1+2+2+2+2+2+2=15$ marks]

15

a) We have $1-j=\sqrt{2}e^{-j\pi/4}$. So

$$(1-j)^{15}=2^7\sqrt{2}e^{-j(15\pi/4}=128\sqrt{2}e^{j(\pi/4)}=128\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}j\right)=128(1+j).$$

[4 marks]

b) Write $z=r\cos\theta+jr\sin\theta$. The polar form of -27 is $27(\cos(\pi)+j\sin(\pi))$. De Moivre's Theorem gives

$$r^3(\cos 3\theta+j\sin 3\theta)=27(\cos(\pi)+j\sin(\pi)).$$

So $r^3=27$, $\cos 3\theta=\cos(\pi)$, $\sin 3\theta=\sin(\pi)$. So $r=3$ and $3\theta=\pi+2n\pi$, any integer n. [6 marks]

Distinct values of z are given by taking $n=0$, 1, 2, that is, $\theta=\pi/3$, π, $5\pi/3$, So the solutions to $z^3=-27$ are

$$z=3\cos(\pi/3)+3j\sin(\pi/3)=(3/2)(1+\sqrt{3}j),\qquad\text{(2 marks)}$$

$$z=3\cos(\pi)+3j\sin(\pi)=-3\qquad\text{(1 mark)}$$

$$z=3\cos(5\pi/3)+3j\sin(5\pi/3)=(3/2)(1-\sqrt{3}j).\qquad\text{(2 marks)}$$

[$5=2+1+2$ marks]
[$4+6+5=15$ marks]