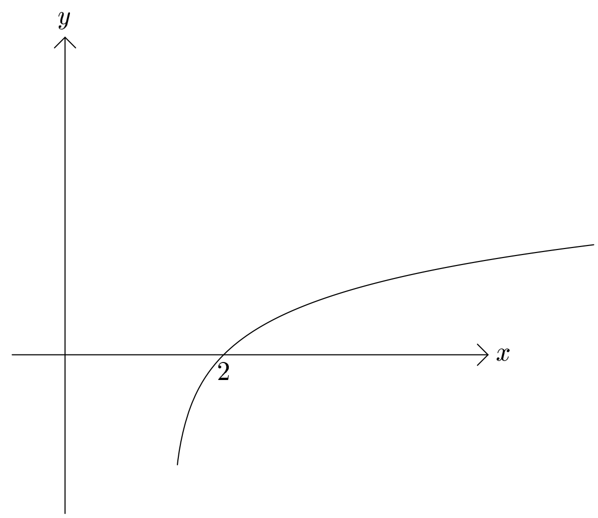
All questions are similar to homework problems.

MATH191 Solutions September 2007
Section A

1. The maximal domain is $(2, \infty)$ and the range is $\mathbb{R}$ (1 mark each).
The graph is shown below (1 mark). It crosses the x-axis at $x = 3$ (1 mark).



[1 + 1 + 1 + 1 = 4 marks]

2. We have $f(0) = \frac{1}{2}$, $f'(x) = -\frac{1}{2}(4+x)^{-3/2}$, so $f'(0) = -1/16$, and $f''(x) = \frac{3}{4}(4+x)^{-5/2}$, so $f''(0) = 3/128$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).

Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = \frac{1}{2} - x/16 + 3x^2/256 + \cdots.$$

(1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = 2 + x/4 - x^2/64$).
[3 + 1 + 1 = 5 marks]

3.

a) $r = \sqrt{2}$ (1 mark). $\theta = 3\pi/4$ (2 marks).

b) $x = 2\cos(\pi) = -2$. $y = 2\sin(\pi) = 0$. (1 mark each)

Subtract one mark for each answer not given exactly.
[3 + 2 = 5 marks]

4.

$$\begin{aligned}\int_1^2 e^{2x} + x^{-1/2}\,dx &= \left[\frac{e^{2x}}{2} + 2x^{1/2}\right]_1^2 \qquad \text{(3 marks)}\\ &= \frac{e^4}{2} - \frac{e^2}{2} + 2(\sqrt{2}-1). \qquad \text{(2 marks)}\end{aligned}$$

$3 + 2 = 5$ marks]

5. Differentiating the equation with respect to x gives

$$3x^2 + 2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0 \quad \text{(2 marks)}.$$

Hence

$$\frac{dy}{dx} = \frac{-3x^2 - 2xy - y^2}{x^2 + 2xy} \quad \text{(2 marks)}.$$

Thus $\frac{dy}{dx}$ is equal to $\frac{-3+2-1}{-1} = 2$ when $(x, y) = (1, -1)$. (2 marks)
The equation of the tangent at this point is therefore

$$y + 1 = 2x - 2,$$

or

$$y = 2x - 3. \qquad \text{(2 marks)}$$

[$2 + 2 + 2 + 2 = 8$ marks]

6.

a) By the product rule and chain rule,

$$\frac{d}{dx}(x^2\cos 2x) = 2x\cos(2x) - 2x^2\sin(2x). \qquad \text{(2 marks)}.$$

b) By the chain rule,

$$\frac{d}{dx}(x^2 + x + 1)^9 = 9(2x+1)(x^2 + x + x1)^8 \qquad \text{(3 marks)}.$$

c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{e^x}{x^2-1}\right) = \frac{e^x(x^2-1) - 2xe^x}{(x^2-1)^2}. \qquad \text{(2 marks)}.$$

[$2 + 3 + 2 = 7$ marks]

7. $f'(x) = -e^{-x} + 1$. Stationary points are given by solutions of $f'(x) = 0$. So there is exactly one stationary point, namely $x = 0$. (3 marks)

To determine its nature, $f''(x) = e^{-x}$: so $f''(0) = 1 > 0$, and 0 is a local minimum. (2 marks).
[3 + 2 = 5 marks]

8.

$$
\begin{aligned}
z_1 + z_2 &= 5 \qquad \text{(1 mark)} \\
z_1 - z_2 &= -1 + 2j \qquad \text{(1 mark)} \\
z_1 z_2 &= (2+j)(3-j) = 6 + j - j^2 = 7 + j \qquad \text{(2 marks)} \\
z_1/z_2 &= \frac{(2+j)(3+j)}{(3-j)(3+j)} = \frac{5+5j}{10} = \frac{1+j}{2} \qquad \text{(2 marks).}
\end{aligned}
$$

[1 + 1 + 2 + 2 = 6 marks]

9. $\sin^{-1}(\frac{-1}{2}) = -\pi/6$ (1 mark)
The general solution of $\sin\theta = \frac{-1}{2}$ is

$$\theta = -\pi/6 + 2n\pi \text{ or } \theta = -5\pi/6 + 2n\pi$$

for any $n \in \mathbb{Z}$ (3 marks)
[1 + 3 = 4 marks]

10.

$$
\begin{aligned}
\mathbf{a} + \mathbf{b} &= 4\mathbf{i} - \mathbf{j} \quad \text{(1 mark)} \\
\mathbf{a} - \mathbf{b} &= 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \quad \text{(1 mark)} \\
|\mathbf{a}| &= \sqrt{3^2 + 1 + 1} = \sqrt{11} \qquad \text{(1 mark)} \\
|\mathbf{b}| &= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \qquad \text{(1 mark)} \\
\mathbf{a} \cdot \mathbf{b} &= 3 - 2 - 1 = 0 \qquad \text{(1 mark).}
\end{aligned}
$$

Hence the angle between **a** and **b** is $\pi/2$ (1 mark).
[1 + 1 + 1 + 1 + 1 + 1 = 6 marks]

Section B

11. The Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{(2 marks)}$$

Hence

a)

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} \cdots \qquad \text{(1 mark)}$$

b)

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{(1 mark)}$$

c)

$$e^x + e^{-x} = 2 + x^2 + 2\frac{x^4}{4!} + \cdots \qquad \text{(2 marks)}$$

d)

$$e^x - e^{-x} = 2x + 2\frac{x^3}{3!} + \cdots \qquad \text{(2 marks)}$$

e)

$$\frac{e^{2x} + 1 - 2e^x}{x^2} = \frac{2 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots - 2 - 2x - x^2 - \frac{x^3}{3} - \frac{x^4}{12} \cdots}{x^2}$$

$$= 1 + x + \frac{7x^2}{12} \cdots \qquad \text{(4 marks)}$$

So $f(0.1) = 1.1 + 0.00583333.. = 1.106$ to 3 decimal places. (3 marks)

[$2 + 1 + 1 + 2 + 2 + 4 + 3 = 15$ marks]

12.

a) The radius of the convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $|a_n/a_{n+1}| = 2(n+1)^2/n^2$, which converges to 2. So the radius of convergence is 2. (4 marks)

At $R = 2$ the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a standard convergent series. At $R = -2$ the series becomes

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}.$$

Since the terms of the series are in modulus the same as for $R = 2$, this series is also convergent. (5 marks)

b) In this case $a_n = 1./2^n\sqrt{n}$, so $|a_n/a_{n+1}| = 2\sqrt{(n+1)/n}$, which tends to 2 as $n \to \infty$. Hence $R = 2$. (4 marks)

At $R = 2$ the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which diverges. (2 marks)

[$4 + 5 + 4 + 2 = 15$ marks]

13. $f'(x) = 3x^2 + 6x = 0 \Leftrightarrow x = 0$ or $x = -2$. Since $f'(x) = 3x(x+2)$, $f'(x) > 0$ if $x < -2$ or $x > 0$, and $f'(x) < 0$ if $x \in (0, 2)$. So f is increasing on each of the intervals $(-\infty, -2)$ and $(0, \infty)$, and decreasing on $(-2, 0)$. So the graph of f can cross the x-axis at most 3 times, and can have at most 3 zeros. We have

$$f(-3) = -1, \quad f(-2) = -8 + 12 - 1 = 3, \quad f(-1) = 1,$$

$$f(-\frac{1}{2} = -\frac{3}{8}, \quad f(0) = -1, \quad f(1) = 3.$$

So f must change sign on each of the intervals $(-3, -2)$, $(-1, -\frac{1}{2})$, $(0, 1)$, that is, have zeros in each of these intervals. (6 marks)

The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n - 1}{3x_n^2 + 6x_n} = \frac{2x_n^3 + 3x_n^2 + 1}{3x_n^2 + 6x_n} \qquad \text{(3 marks)}$$

Hence

$$x_1 = \frac{2x_0^3 + 3x_0^2 + 1}{3x_0^2 + 6x_0} = \frac{2}{15/4} = \frac{8}{15} = 0.5333333..$$

So $x_1 = 0.533333$ to 6 decimal places. (1 mark)

$$x_2 = \frac{2x_1^3 + 3x_1^2 + 1}{3x_1^2 + 6x_1} = \frac{1024 + 45 \times 64 + 15 \times 225}{15 \times (192 + 48 \times 15)} = \frac{7279}{13680} = 0.5320906..$$

So $x_2 = 0.532091$ to 6 decimal places. (2 marks)

$$f(x_2) = 0.000007102..$$

So $f(x_2) = 0.000007$ to 6 decimal places. (3 marks)
[$6 + 3 + 1 + 2 + 3 = 15$ marks]

14. For horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{1+x} = 0,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(\frac{3}{1+x} + 1 - x\right) = 1 - \infty = -\infty.$$

So $y = 0$ is a horizontal asymptote (although only at $-\infty$). (2 marks)

For vertical asymptotes: the only possible asymptote is where $x + 1 = 0$, that is, where $x = -1$. We have

$$\lim_{x\to(-1)-} f(x) = \lim_{x\to(-1)-} \frac{1}{1+x} = -\infty$$

So $x = -1$ is a vertical asymptote, although

$$\lim_{x\to(-1)+} f(x) = \lim_{x\to(-1)+} (1 + x/2) = \frac{1}{2}. \qquad \text{(2 marks)}$$

For points of continuity: the only possible discontinuities are at -1 and 0. -1 is certainly a discontinuity, because it is a vertical asymptote. 0 is a point of continuity, because

$$\lim_{x\to1-} f(x) = \lim_{x\to1+} f(x) = \frac{3}{2}. \qquad \text{(2 marks)}$$

For $x \in (1, \infty)$ we can write

$$f(x) = \frac{4 - x^2}{1+x} = \frac{3}{1+x} + 1 - x.$$

So we have

$$f'(x) = \begin{cases} -\frac{1}{(1+x)^2} & \text{if } x \in (-\infty, -1) \\ \frac{1}{2} & \text{if } x \in (-1, 1) \\ -1 - \frac{3}{(1+x)^2} & \text{if } x \in (1, \infty) \end{cases} \qquad \text{(2 marks)}$$

The function is not differentiable at -1 because it is not continuous there. The only other point that needs checking is 1. There the function is not differentiable because the left derivative is $\frac{1}{2}$ and from the formula for $f'(x)$ above, the right derivative is $-\frac{7}{4}$. (2 marks)

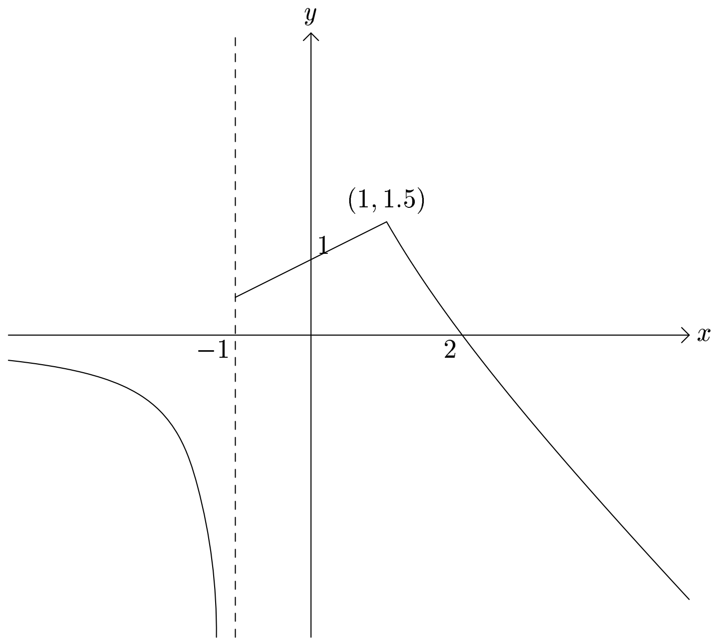
So

$$f'(x) \begin{cases} < 0 & \text{if } x \in (-\infty, -1) \\ > 0 & \text{if } x \in (-1, 1) \\ < 0 & \text{if } x \in (1, \infty) \end{cases}$$

So there are no stationary points. The function f is decreasing on each of the intervals $(-\infty, -1)$ and $[1, \infty)$ and increasing on $[-1, 1]$. (2 marks)

For zeros: since $\frac{1}{1+x} \neq 0$ for any x, and $1 + \frac{1}{2}x > 0$ for $x \in [-1, 1]$, the only possible zero is on $(1, \infty)$, when $4 - x^2 = 0$, that is, when $x = 2$. (1 mark)

The graph of f is as shown.



[2 marks]

[2 + 2 + 2 + 2 + 2 + 2 + 1 + 2 = 15 marks]

15 Write $z = r(\cos\theta + j\sin\theta)$.
a) The polar form of $-4j$ is $4(\cos(-\pi/2) + j\sin(-\pi/2))$. De Moivre's Theorem gives

$$r^2(\cos 2\theta + j\sin 2\theta) = 4(\cos(-\pi/2) + j\sin(-\pi/2)).$$

So $r^2 = 4$, $\cos 2\theta = \cos(-\pi/2)$,$\sin 2\theta = \sin(-\pi/2)$. So $r = 2$ and $2\theta = -\pi/2 + 2n\pi$, any integer n. So $\theta = -\pi/4 + n\pi$, any integer n. So the possible values for z are

$$z = 2(\cos(-\pi/4) + j\sin(-\pi/4)) = \sqrt{2} - \sqrt{2}j$$

and

$$z = 2(\cos(3\pi/4) + j\sin(3\pi/4)) = -\sqrt{2} + \sqrt{2}j.$$

[6 marks]

b) The polar form of j is $\cos(\pi/2) + j\sin(\pi/2)$. De Moivre's Theorem gives

$$r^3(\cos 3\theta + j\sin 3\theta) = \cos(\pi/2) + j\sin(\pi/2).$$

So $r^3 = 1$, $\cos 3\theta = \cos(\pi/2)$,$\sin 3\theta = \sin(\pi/2)$. So $r = 1$ and $3\theta = \pi/2 + 2n\pi$, any integer n. Distinct values of z are given by taking $n = 0$, 1, 2, that is, $\theta = \pi/6$, $5\pi/6$, $3\pi/2$. So the solutions to $z^3 = j$ are

$$z = \cos(\pi/6) + j\sin(\pi/6) = \frac{1}{2}(\sqrt{3} + j),$$

$$z = \cos(5\pi/6) + j\sin(5\pi/6) = \frac{1}{2}(-\sqrt{3} + j)$$

,

$$z = \cos(3\pi/2) + j\sin(3\pi/2) = -j.$$

[9 marks]
[6 + 9 = 15 marks]