All questions are similar to homework problems.

MATH191 Solutions January 2008
Section A

1. To find the inverse function,

$$y = \frac{x+3}{x-2} \;\Leftrightarrow\; y(x-2) = x+3 \;\Leftrightarrow yx - 2y = x+3$$

$$\Leftrightarrow x(y-1) = 2y+3 \;\Leftrightarrow x = \frac{2y+3}{y-1}.$$

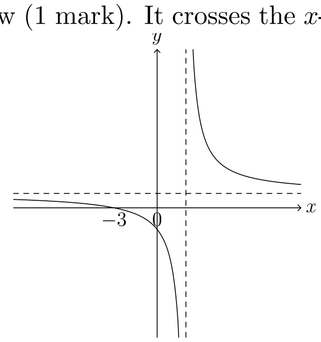
(1 mark)

So the inverse function is given by

$$f^{-1}(y) = \frac{2y+3}{y-1} \quad \text{or} \quad f^{-1}(x) = \frac{2x+3}{x-1}.$$

(1 mark)

The graph is shown below (1 mark). It crosses the x-axis at $x = -3$ and the



y-axis at $y = -\frac{3}{2}$. (1 mark).
$[1+1+1+1 = 4$ marks]

2. We have $f(0) = 1$, $f'(x) = 2(1-2x)^{-2}$ and $f''(x) = 8(1-2x)^{-3}$, so $f'(0) = 2$, and $f''(0) = 8$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).

Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = 1 + 2x + (8/2)x^2 + \cdots = 1 + 2x + 4x^2 + \cdots$$

(1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = 1 + 2x + 4x^2$).
$[3+1+1 = 5$ marks]

3.

a) $r = 4\sqrt{2}$ (1 mark). $\theta = \frac{5\pi}{4}$ (2 marks).

b) $x = 4\sqrt{2}\cos(5\pi/4) = 4\sqrt{2}(-1/\sqrt{2}) = -4$. $y = 4\sqrt{2}\sin(5\pi/4) = 4\sqrt{2}(-1/\sqrt{2}) = -4$. (1 mark each)

Subtract one mark for each answer not given exactly. It is OK to say that from
a) $x = -4$ and $y - -4$
[$3 + 2 = 5$ marks]

4.

$$\int_0^1 \left((1+x)^{-1} + (1+x)^{-2}\right) dx = \left[\ln(1+x) - (1+x)^{-1}\right]_0^1 \qquad \text{(3 marks)}$$
$$= (\ln(2) - \ln(1)) - \frac{1}{2} + 1)$$
$$= \ln 2 + \frac{1}{2}. \qquad \text{(2 marks)}$$

[$3 + 2 = 5$ marks]

5. Differentiating the equation with respect to x gives

$$3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \quad \text{(2 marks)}.$$

Hence

$$\frac{dy}{dx} = \frac{-3x^2 - y}{x + 3y^2} \quad \text{(2 marks)}.$$

Thus $\frac{dy}{dx}$ is equal to $-\frac{1}{2}$ when $(x, y) = (1, -1)$. (2 marks).
The equation of the tangent at this point is therefore

$$y + 1 = -\frac{1}{2}(x - 1) \qquad \text{or } y = -\frac{1}{2}(x + 1) \qquad \text{(2 marks)}.$$

[$2 + 2 + 2 + 2 = 8$ marks]

6.

a) By the chain rule,

$$\frac{d}{dx}(\sin(x^3 - 1)) = 3x^2 \cos(x^3 - 1). \qquad \text{(2 marks)}.$$

b) By the quotient rule,

$$\frac{d}{dx}\left(\frac{\sin x}{x^2 + 1}\right) = \frac{(x^2 + 1)\cos x - 2x \sin x}{(x^2 + 1)^2} \qquad \text{(2 marks)}.$$

c) By the chain rule,

$$\frac{d}{dx}(\ln(x^3 + 2x - 1) = \frac{3x^2 + 2}{x^3 + 2x - 1}. \qquad \text{(2 marks)}.$$

[2 + 2 + 2 = 6 marks]

7.

$$f'(x) = -2x(x^2+1)^{-2}.$$

Stationary points are given by solutions of $f'(x) = 0$, So there is exactly one stationary point, namely $x = 0$. (3 marks)

To determine its nature,

$$f''(x) = -2(x^2+1)^{-2} + 8x^2(x^2+1)^{-3}.$$

So $f''(0) = -2 < 0$, and 0 is a local maximum. (3 marks)
[3 + 3 = 6 marks]

8.

$$\begin{aligned} z_1 + z_2 &= 2 - j \qquad \text{(1 mark)} \\ z_1 - z_2 &= 5j \qquad \text{(1 mark)} \\ z_1 z_2 &= (1+2j)(1-3j) = 1 - j - 6j^2 = 7 - j \qquad \text{(2 marks)} \\ z_1/z_2 &= \frac{(1+2j)(1+3j)}{(1-3j)(1+3j)} = \frac{1+5j+6j^2}{10} = \frac{-1+j}{2} \qquad \text{(2 marks).} \end{aligned}$$

[1 + 1 + 2 + 2 = 6 marks]

9. $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$ (1 mark)

The general solution of $\sin\theta = \frac{-\sqrt{3}}{2}$ is

$$\theta = -\pi/3 + 2n\pi \text{ or } \theta = -2\pi/3 + 2n\pi$$

for any $n \in \mathbb{Z}$. (3 marks)
[1 + 3 = 4 marks]

10.

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= 4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \quad \text{(1 mark)} \\ \mathbf{a} - \mathbf{b} &= 2\mathbf{i} - 6\mathbf{j} + \mathbf{k} \quad \text{(1 mark)} \\ |\mathbf{a}| &= \sqrt{3^2+1^2+2^2} = \sqrt{14} \qquad \text{(1 mark)} \\ |\mathbf{b}| &= \sqrt{1^2+5^2+1^2} = \sqrt{27} \qquad \text{(1 mark)} \\ \mathbf{a}\cdot\mathbf{b} &= 2 - 5 + 3 = 0 \qquad \text{(1 mark).} \end{aligned}$$

Hence the angle between **a** and **b** is $\pi/2$ (1 mark).
[1 + 1 + 1 + 1 + 1 + 1 = 6 marks]

Section B

11. The Maclaurin series expansion of $(1+x)^{-1}$ is

$$= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots\cdots \qquad \text{(2 marks)}$$

Hence the other Maclaurin series are:

a) for $(1+2x)^{-1}$,

$$1 - 2x + 4x^2 \cdots + (-1)^n 2^n x^n \cdots \qquad \text{(2 marks)}$$

b) for $(2+x)^{-1} = 2^{-1}(1+x/2)^{-1}$,

$$\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 + \cdots + (-1)^n \frac{1}{2^{n+1}} x^{n+1} + \cdots \qquad \text{(3 marks)}$$

c) for $(1+x^2)^{-1}$,

$$1 - x^2 + x^4 + \cdots (-1)^n x^{2n} + \cdots \qquad \text{(3 marks)}$$

d) The Maclaurin series of $g(x) = (1+x^2)^{-1}$ is

$$g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \cdots + \frac{g^n(0)}{n!}x^n + \cdots$$

So comparing coefficients of x^n we see that

$$g^{(n)}(0) = \begin{matrix} 0 & \text{if } n \text{ is odd,} \\ (-1)^n n! & \text{if } n \text{ is even.} \end{matrix} \qquad \text{(5 marks)}$$

[2 + 2 + 3 + 3 + 5 = 15 marks]

12.

a) The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $|a_n/a_{n+1}| = (n+1)/(n+2)$, which converges to 1. So the radius of convergence is 1. (4 marks)

At $R = 1$ the series becomes

$$\sum_{n=0}^{\infty} (n+1)$$

which is divergent as the terms are not tending to 0. At $R = -1$ the series becomes

$$\sum_{n=1}^{\infty} (-1)^n (n+1).$$

which again diverges as the terms are not converging to 0 (2 marks)

b) In this case $a_n = 4^{-n}(n+1)^{-1}$, so $|a_n/a_{n+1}| = 4(n+2)/(n+1)$, which tends to 4 as $n \to \infty$. Hence $R = 4$. (4 marks)

At $R = 4$ the series becomes

$$\sum_{n=0}^{\infty} (n+1)^{-1} = \sum_{n=1}^{\infty} n^{-1}$$

which diverges. At $R = -4$ the series becomes

$$\sum_{n=0}^{\infty} (-1)^n (n+1)^{-1} = \sum_{n=1}^{\infty} (-1)^n n^{-1}$$

which converges as it is an alternating series and $1/n$ decreases to 0 as $n \to \infty$ (5 marks)

$[4 + 2 + 4 + 5 = 15$ marks$]$

13. For $f(x) = x^3 + 2x^2 + x - 2$, $f'(x) = 3x^2 + 4x + 1 = (3x+1)(x+1) = 0 \Leftrightarrow x = -1$ or $x = -1/3$. Now $3x + 1$ and $x + 1$ have the same sign if $x < -1$ or $x > -1/3$, and opposite signs if $-1 < x < -1/3$. So f is increasing on each of the intervals $(-\infty, -1)$ and $(-1/3, \infty)$, and decreasing on $(-1, -1/3)$. We have

$$f(-1) = -2, \quad f(-1/3) = -58/27, \quad f(0) = -2, \quad f(1) = 2.$$

So $f < 0$ on $(-\infty, 0)$, $f > 0$ on $(1, \infty)$ and there is just one zero of f, which is in $(0, 1)$. (6 marks)

The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 + 2x_n^2 + x_n - 2}{3x_n^2 + 4x_n + 1} = \qquad \text{(3 marks)}$$

Hence

$$x_1 = 1 - \frac{2}{8} = \frac{3}{4}, \quad f(x_1) = 0.296875 \qquad \text{(1 mark)}$$

$$x_2 = 0.69780220, \qquad f(x_2) = 0.01437376 \qquad \text{(2 mark)}$$

$$x_3 = 0.69562448, \qquad f(x_3) = 0.00002 \qquad \text{(3 marks)}$$

$[6 + 3 + 1 + 2 + 3 = 15$ marks $]$

14. For horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{2-x} = 0,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(x + (2-x)^{-1}\right) = +\infty.$$

So $y = 0$ is a horizontal asymptote (although only at $-\infty$). (2 marks)

For vertical asymptotes: the only possible asymptote is where $x - 2 = 0$, that is, where $x = 2$. We have

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \left(x + \frac{1}{2}\right) = \frac{5}{2}, \quad \lim_{x\to 2+} \left(x + \frac{1}{2-x}\right) = -\infty$$

So $x = 2$ is a vertical asymptote. (1 mark)

For points of continuity: the only possible discontinuities are at 2 and 0. 2 is certainly a discontinuity, because it is a vertical asymptote. 0 is not a discontinuity because $f(0-) = \frac{1}{2} = f(0+) = f(0)$. (2 marks)

We have

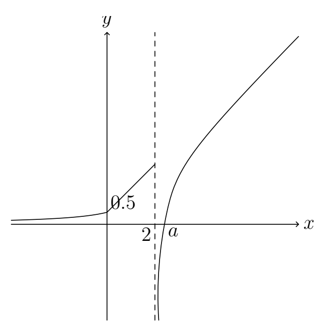
$$f'(x) = \begin{cases} (2-x)^{-2} & \text{if } x \in (-\infty, 0), \\ 1 & \text{if } x \in (0, 2), \\ 1 + (2-x)^{-2} & \text{if } x \in (2, \infty), \end{cases}$$

[2 marks]

The function is not differentiable at 2 because it is not continuous there. The only other point that needs checking is 0. There the function is not differentiable because the left derivative is $\frac{1}{4}$ and the right derivative is 1. (2 marks)

Now $f'(x) > 0$ on each of the intervals $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$ So there are no stationary points. The function f is increasing on each of the intervals $(-\infty, 0]$, $[0, 2)$ and $(2, \infty)$ (2 marks)

For zeros: since $(2-x)^{-1} \neq 0$ for any x, and $x + \frac{1}{2} > 0$ for $x \in (0, 2)$. So the only possible zero is on $(2, \infty)$, when $x + (2-x)^{-1} = 0$, that is, when $2x - x^2 + 1 = 0$ and $x > 2$, that is when $x^2 - 2x - 1 = 0$ and $x > 2$. The zeros of $x^2 - 2x - 1$ are $1 \pm \sqrt{2}$. Since $1 + \sqrt{2} > 2$, and $1 - \sqrt{2} < 2$, the unique zero of f is $1 + \sqrt{2}$ (2 mark)



The graph of f is as shown, wth $a = 1 + \sqrt{2}$. [2 marks]

$[2 + 1 + 2 + 2 + 2 + 2 + 2 + 2 = 15$ marks]

15

a) We have $1+j = \sqrt{2}e^{j\pi/4}$. So

$$(1+j)^{27} = 2^{13}\sqrt{2}e^{j(27\pi/4)} = 8192\sqrt{2}e^{j(3\pi/4)} = 8192\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = 8192(-1+j).$$

[4 marks]

b) Write $z = r\cos\theta + jr\sin\theta$.The polar form of -16 is $16(\cos(\pi) + j\sin(\pi))$. De Moivre's Theorem gives

$$r^4(\cos 4\theta + j\sin 4\theta) = 16(\cos(\pi) + j\sin(\pi)).$$

So $r^4 = 16$, $\cos 4\theta = \cos(\pi)$, $\sin 4\theta = \sin(\pi))$. So $r = 2$ and $4\theta = \pi + 2n\pi$, any integer n. [7 marks]

Distinct values of z are given by taking $n = 0$, 1, 2, 3 that is, $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$. So the solutions to $z^4 = -16$ are

$$z = 2\cos(\pi/4) + 2j\sin(\pi/4) = \sqrt{2}(1+j),$$

$$z = 2\cos(3\pi/4) + 2j\sin(3\pi/4) = \sqrt{2}(-1+j)$$

$$z = 2\cos(5\pi/4) + 2j\sin(5\pi/4) = \sqrt{2}(-1-j)$$

$$z = 2\cos(7\pi/4) + 2j\sin(7pi/4) = \sqrt{2}(1-j)$$

[4 marks]

[$4+7+4 = 15$ marks]