SECTION A

1. State the maximal domain and range of the function

$$f(x) = \sin x.$$

Sketch the graph of $y = f(x)$ for $-2\pi \leq x \leq 2\pi$, indicating clearly the coordinates of any points where the graph crosses the axes.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = \ln(1 + 3x)$$

up to and including the term in x^2.

[5 marks]

3.

a) Convert $(x, y) = (-2, -2)$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (1, 5\pi/6)$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$, etc.

[6 marks]

4. Calculate the integral

$$\int_1^4 \left(e^{-x} - \frac{1}{x}\right) \mathrm{d}x,$$

evaluating the result to three decimal places.

[5 marks]

5. Consider the curve defined by

$$x^2 + 2xy + y^3 = 1.$$

Find an expression for $\frac{dy}{dx}$ in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1, 0)$. [8 marks]

6. Differentiate the following functions with respect to x:

$$\text{a) } \cos(x^2+x); \qquad \text{b) } (x^2+x)\cos x; \qquad \text{c) } \frac{\sin x}{x}.$$

[6 marks]

7. Show that the function

$$f(x) = x^2 - \ln(x^2) \qquad (x \neq 0)$$

has exactly two stationary points. Determine whether each of these stationary points is a local maximum, a local minimum, or a point of inflection.

[5 marks]

8. Let z_1 and z_2 be the complex numbers given by $z_1 = 3+2j$ and $z_2 = 1-j$. Calculate z_1+z_2, z_1-z_2, z_1z_2, and z_1/z_2.

[6 marks]

9. State the value of $\cos^{-1}(\sqrt{3}/2)$ (you should give an exact answer in radians). Give the general solution of the equation

$$\cos\theta = \frac{\sqrt{3}}{2}.$$

[4 marks]

10. Let $\mathbf{a} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. Find $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a}\cdot\mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$ (in radians, to 3 decimal places)?

[6 marks]

SECTION B

11. Give the Maclaurin series expansion of the function $f(x) = \sin x$ up to and including the term in x^5 (you are not required to show any working if you remember this expansion).

[2 marks]

Hence, or otherwise, give the first three non-zero terms of the Maclaurin series expansions of the following functions:

a) $x^3 \sin x$; b) $\sin(x^3)$; c) $\sin(2x)$; d) $\sin^2 x$.

[13 marks]

12. Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{1}{n\,4^n} x^n.$$

[8 marks]

Use the alternating series test to show that the series converges when $x = -R$. Write down the series when $x = R$, and state whether it is convergent or divergent. Hence state all of the (real) values of x for which the power series converges.

[7 marks]

13. Sketch the graphs of $y = x^3$ and $y = \cos x$ on the same axes, and explain carefully why the equation

$$f(x) = x^3 - \cos x = 0$$

has exactly one solution (for real x), which lies between $x = 0$ and $x = \frac{\pi}{2}$.

[9 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 1$ to obtain successive approximations x_1, x_2, and x_3 to a solution of the above equation $f(x) = 0$: give each approximation to 6 decimal places.

[6 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + x + 1 & \text{if } x \leq 0 \\ \frac{1}{1-x} & \text{if } x > 0. \end{cases}$$

Sketch the graph of $y = f(x)$, indicating clearly the positions of any zeros, stationary points, and asymptotes. (You will get no marks for sketching the graph unless you show how you have determined the positions of these features.)

[12 marks]

At which value or values of x is $f(x)$ not continuous? At which value or values is it not differentiable?

[3 marks]

15.

By using de Moivre's theorem, find the integers a, b, c, d, and e such that

$$\cos 4\theta = a\cos^4\theta + b\cos^2\theta\sin^2\theta + c\sin^4\theta$$

and

$$\sin 4\theta = d\cos^3\theta\sin\theta + e\cos\theta\sin^3\theta.$$

[10 marks]

Check both of these results when $\theta = \frac{\pi}{4}$ (you should work with exact values of $\cos\theta$ and $\sin\theta$, rather than evaluating them on your calculator).

[5 marks]