UNIVERSITY OF LIVERPOOL

MATH191

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Three hours

Full marks are obtained by complete answers to all of Section A and THREE questions from Section B. The best three answers to Section B will be taken into account, but all answers will be marked. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks and each question in Section B is worth 15% of the available marks.

SECTION A

1. Find the inverse function of

$$f(x) = \frac{1+2x}{1-x}.$$

Determine the domain and range of f. Sketch the graph of f, marking clearly any vertical or horizontal asymptotes. [7 marks]

2.

a) Convert $(r, \theta) = (3, -3\pi/4)$ from polar to Cartesian coordinates.

b) Convert $(x, y) = (-2, 2)$ from Cartesian to polar coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$ etc..

[5 marks]

3. Compute the following. (It may be that the answer is $+\infty$ or $-\infty$ in some cases.)

$$\text{a) } \lim_{x \to (-1)-} \frac{x^2+2}{x+1}, \qquad \text{b) } \lim_{x \to \pm\infty} \frac{2x^2-x-1}{x^2-2x+2}$$

$$\text{c) } \lim_{x \to -1} \frac{3x^2+x-2}{4x^2+5x+1}, \qquad \text{d) } \lim_{x \to 0} \frac{\tan x}{x^2+x}$$

Hint: It may be that the answer is $+\infty$ or $-\infty$ in some cases. You may use l'Hôpital's Rule where it is appropriate to do so.

[8 marks]

4. Differentiate the following functions with respect to x:

$$\text{a) } 3x \sin x; \qquad \text{b) } \frac{x+1}{x^2+3}; \qquad \text{c) } \sin(1/x) \qquad \text{d) } \cos(\sin x).$$

[8 marks]

5. Evaluate the definite integral

$$\int_0^{\pi/6} (\cos(3x) - \sin^2(2x))\mathsf{d}x,$$

giving your answer exactly.

[6 marks]

6. Consider the curve defined by

$$4xy - y^3 = 0.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1, 2)$.

[8 marks]

7. Let z_1 and z_2 be the complex numbers given by $z_1 = -2+3j$ and $z_2 = 1-4j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

8. Let $\mathbf{a} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i}-\mathbf{j}-\mathbf{k}$. Find $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a}\cdot\mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[7 marks]

SECTION B

9. a) Write down the Maclaurin series expansion of $\dfrac{1}{1-x}$, including the general term (you are not required to show any working if you remember the expansion).

[3 marks]

b) Hence, or otherwise, determine the Maclaurin series of the following, including the general term.

$$\text{(i) } \frac{1}{1+3x} \qquad \text{(ii) } \frac{1}{1-x^3}.$$

[4 marks]

c) Find the Maclaurin series of $\frac{1}{(1-x)^2}$, including the general term. Once again, you are not required to show any working if you remember the expansion

[5 marks]

d) Find the radius of convergence of the Maclaurin series of $\frac{1}{(1-x)^2}$.

[3 marks]

10. a) Show that the stationary points of

$$f(x) = x + \sin x$$

are all points of the form $(2n+1)\pi$, $n \in \mathbb{Z}$. Determine which (if any) of these stationary points are maxima and which are minima, and which are points of inflection. Determine whether the function f is increasing or decreasing or neither.

[6 marks]

b) Now find all the stationary points of

$$g(x) = x + 2\sin x.$$

Determine which of the stationary points are maxima and which are minima. Sketch the graph of g.

[9 marks]

11. a) Use calculus to find the intervals on which the function $f(x) = x^3 - 7x + 3$ is strictly increasing and strictly decreasing. Show also that the function has three zeros, one in $(-3, -2)$, one in $(0, 1)$ and one in $(2, 3)$.

[7 marks]

b) Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 0$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x) = 0$ in $(0, 1)$. Give x_1, x_2 and x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[8 marks]

12. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \dfrac{4}{x^2+1} \text{ if } x \in (-\infty, 1), \\ \\ x^2 + 1 \text{ if } x \in [1, \infty). \end{cases}$$

Find any horizontal and vertical asymptotes of f. Calculate $f'(x)$ where it exists. Determine whether f is continuous or not, and whether it is differentiable or not, at $x = 1$, and give the left and right derivatives at this point, if they exist. Determine any stationary points of f and their types. Sketch the graph of f.

[15 marks]

13. Find all solutions z of each of the following equations, in the form $z = a + bj$ for real a and b. In each case, indicate where the solutions are in the plane.

a) $jz^2 + 3z + 4j = 0$

[5 marks]

b) $z^4 = -1 + \sqrt{3}j$

[10 marks]

Hint: You might find de Moivre's Theorem useful in the second part.