UNIVERSITY OF LIVERPOOL

MATH191

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Three hours

Full marks are obtained by complete answers to all of Section A and THREE questions from Section B. The best three answers to Section B will be taken into account, but all answers will be marked. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks and each question in Section B is worth 15% of the available marks.

SECTION A

1. Find the inverse function of

$$f(x) = \frac{3x+5}{x-1}.$$

Determine the domain and range of f. [5 marks]

2.

a) Convert $(r, \theta) = (2, -\pi/6)$ from polar to Cartesian coordinates.

b) Convert $(x, y) = (-\sqrt{3}, -1)$ from Cartesian to polar coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{3}$ etc..

[5 marks]

3. Let $f(x)$ be as in question 1. Compute the following. (It may be that the answer is $+\infty$ or $-\infty$.)

$$\text{a)} \lim_{x\to 1-} f(x), \qquad \text{b)} \lim_{x\to\infty} f(x).$$

Sketch the graph of f

[5 marks]

4. Using l'Hopital's rule or otherwise, compute

$$\text{a)} \lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}, \qquad \text{b)} \lim_{x\to\pi/2} \frac{\cos x}{2x-\pi}.$$

[4 marks]

5. Differentiate the following functions with respect to x:

$$\text{a)}\ x^2e^x; \qquad \text{b)}\ \frac{2x-1}{x^2+4}; \qquad \text{c)}\ (1+\sin x)^{10}.$$

[6 marks]

6. For f as in 5a), that is, $f(x) = x^2e^x$, show that f has exactly two stationary points, and determine the type of each one: that is, local minimum, local maximum or point of inflexion. [5 marks]

7. Evaluate the definite integral

$$\int_0^{\pi/8} (\sqrt{1+2x} - \sin 2x \cos 2x)\mathrm{d}x,$$

giving your answer exactly.

[5 marks]

8. Consider the curve defined by

$$2x^2y^2 - 3xy = 2.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (2, 1)$.

[8 marks]

9. Let z_1 and z_2 be the complex numbers given by $z_1 = 3 - j$ and $z_2 = 2 + 5j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

10. Let $\mathbf{a} = \mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[6 marks]

SECTION B

11. a) Write down the Maclaurin series expansion of $\cos x$, including the general term (you are not required to show any working if you remember the expansion).

[3 marks]

b) Hence, or otherwise, determine the Maclaurin series of

(i) $\cos(2x)$ (ii) $\cos(2x^2)$ (iii) $\cos\sqrt{x}$.

[6 marks]

c) Hence or otherwise, determine

$$\lim_{x\to 0} \frac{4\cos x - \cos 2x - 3\cos(2x^2)}{x^4}.$$

[3 marks]

d) Find the radius of convergence of the Maclaurin series of $\cos\sqrt{x}$.

[3 marks]

12.

a) Find all solutions of the equation

$$\cos\theta = \frac{1}{2}.$$

[3 marks]

b) Using the trigonometric formula

$$\cos(A+B) = \cos A\cos B - \sin A\sin B,$$

find a relationship between $\cos(\theta+\pi/2)$ and $\sin\theta$. [3 marks]

c) Find all solutions of

$$2\cos\theta + 4\sin\theta = 3.$$

[6 marks]

d) Explain why the equation

$$2\cos\theta + 4\sin\theta = 5$$

has no solutions. [3 marks]

[15 marks]

13.

a) Use calculus to show that the function $f(x) = x^3 + 3x - 6$, has exactly one zero and that this zero is in $(1, 2)$. Sketch the graph. Determine whether f is

(i) even or odd or neither;

(ii) increasing or decreasing or neither.

[7 marks]

b) Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 1$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x) = 0$ in $(0, 1)$. Give x_1, x_2 and x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[8 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{x+1} \text{ if } x \in (-\infty, -1), \\ \\ |x| \text{ if } x \in [-1, 1], \\ \\ \dfrac{x^2+1}{2} \text{ if } x \in (1, \infty). \end{cases}$$

Find any horizontal and vertical asymptotes of f. Calculate $f'(x)$ where it exists. Determine whether f is continuous or not, and whether it is differentiable or not, at each of the points $x = -1$, 0 and 1. Give the left and right derivatives at any of these three points where they exist. Show that f has no stationary points but that it does have a local minimum. Sketch the graph of f.

[15 marks]

15.

Find all solutions z of each of the following equations, in the form $z = a + bj$ for real a and b. In each case, indicate where the solutions are in the plane.

a) $z^2 + 3jz + 4 = 0$

[5 marks]

b) $z^3 = 27j$

[10 marks]

Hint: You might find de Moivre's Theorem useful in the second part.