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SECTION A

1. State the maximal domain and range of the function

$$f(x) = \ln(x+1).$$

Sketch the graph of $y = f(x)$, indicating the coordinates of any points where the graph crosses the axes.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = \sqrt{x+4}$$

up to and including the term in x^2.

[5 marks]

3.

a) Convert $(x, y) = (-5, 0)$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (2, 3\pi/4)$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$ etc..

[5 marks]

4. Evaluate the definite integral

$$\int_1^2 \left(e^{-x} + \sqrt{x}\right)\,\mathrm{d}x,$$

giving your answer exactly in terms of e and $\sqrt{2}$.

[5 marks]

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5. Consider the curve defined by

$$x^2 + \cos y = 1.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1/\sqrt{2}, \pi/3)$.

[8 marks]

6. Differentiate the following functions with respect to x:

$$\text{a) } x^2 \sin 2x; \qquad \text{b) } (x^2 + x - 1)^{10}; \qquad \text{c) } \frac{e^x}{x^2+1}.$$

[7 marks]

7. Show that the function

$$f(x) = e^x - x$$

has exactly one stationary point. Determine whether the stationary point is a local maximum, a local minimum, or a point of inflection.

[5 marks]

8. Let z_1 and z_2 be the complex numbers given by $z_1 = 3 + j$ and $z_2 = 2 - j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

9. State the value of $\sin^{-1}(-1\sqrt{2})$ (you should give an exact answer in radians, in terms of π). Give the general solution of the equation

$$\sin\theta = \frac{-1}{\sqrt{2}}.$$

[4 marks]

10. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 5\mathbf{j} + \mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[6 marks]

SECTION B

11. Give the Maclaurin series expansion of the function $f(x) = e^x$ up to and including the term in x^4. (You are not required to show any working if you remember this expansion.)

[2 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions, up to and including the terms in x^4:

a) $e^x - 1$; b) xe^x; c) e^{-x}; d) e^{x^2}.

[10 marks]

Use the Maclaurin series expansion of e^x up to the term in x^4 to obtain an approximation to $f(0.1)$, where

$$f(x) = \frac{e^x - 1}{x} - e^x.$$

You should give your approximation to 4 decimal places.

[3 marks]

12. In each of the following cases, calculate the radius of convergence R of each of the power series, and determine whether the series converges at $x = R$ and at $x = -R$ in a), and at $x = R$ in b).

a) $\sum_{n=1}^{\infty} \frac{1}{2n^2} x^n$; b) $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} x^n$.

[15 marks]

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13. Use calculus to show that the function $f(x) = x^3 - 3x + 1$ has exactly three zeros, one in $(-2, -1)$, one in $(0, 1/2)$ and one in $(1, 2)$.

[9 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = \frac{1}{2}$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x) = 0$ in $(0, \frac{1}{2})$. Give x_2, x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[6 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \frac{1}{1+x} & \text{if } x \in (-\infty, -1) \\ e^x & \text{if } x \in [-1, 0] \\ \frac{1-\frac{1}{4}x^2}{1+x} & \text{if } x \in (0, \infty) \end{cases}$$

Find any horizontal and vertical asymptotes of f. Identify any points of discontinuity of f. Calculate $f'(x)$ where it exists. Identify any points where f is not differentiable. Show that f has no stationary points, and determine maximal intervals on which f is decreasing and maximal intervals on which f is increasing. Find any zeros of f. Sketch the graph of f.

[15 marks]

15. Find all z in the form $a + bj$, for real a and b, such that

$$\text{a) } z^2 = 4j; \qquad \text{b) } z^3 = -8j.$$

Hint: You might find de Moivre's Theorem useful. [15 marks]