Math191 Practice Class Test 2009— Solutions

1. State the domain and range of the following functions:

a) $f(x) = 1+\sin(2x)$ **Solution.** The domain is $\mathbb{R}$ and the range is $[0, 2]$, because the range of $\sin(2x)$ is $[-1, 1]$. [3 marks]

b) $f(x) = |x| + 2$ **Solution.** The domain is $\mathbb{R}$ and the range is $[2, \infty)$ because the range of $|x|$ is $[0, \infty)$. [3 marks]

2. Let

$$f(x) = \frac{x-3}{x+1}.$$

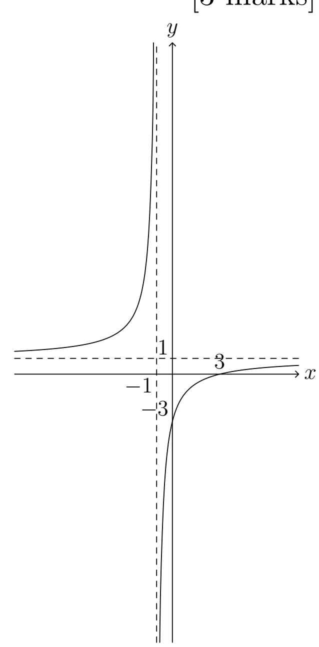
Find the inverse function $f^{-1}(x)$. State the domain and range of f (NOT the inverse function) and sketch the graph, marking any horizontal or vertical asymptotes, and any zeros.

Solution. Set $y = f(x)$ and solve for x in terms of y:

$$y = \frac{x-3}{x+1} \quad \Leftrightarrow \quad (x+1)y = x-3 \quad \Leftrightarrow \quad y+3 = x(1-y) \quad \Leftrightarrow \quad x = \frac{y+3}{1-y}$$

[3 marks]

The domain of f is $(-\infty, -1) \cup (-1, \infty)$. The range of f is the domain of f^{-1}, that is, $(-\infty, 1) \cup (1, \infty)$. [3 marks]



The graph is as shown. [4 marks]

3.

a) Find the exact value of $\sin^{-1}(-\frac{1}{2})$. **Solution.** $\sin^{-1}(-\dfrac{1}{2}) = -\dfrac{\pi}{6}$ [2 marks]

b) Give the general solution of the equation

$$\sin x = -\frac{1}{2}.$$

Solution The general solution is

$$(-1)^{n+1}\frac{\pi}{6} + n\pi.$$

[4 marks]

4. In this question, full marks will only be awarded for *exact* answers (in terms of π, $\sqrt{3}$ etc.) and not for approximations to any number of decimal places.

a) Convert $(2, 5\pi/3)$ from polar to Cartesian coordinates.

Solution.

$$\begin{aligned} x = r\cos\theta &= 2\cos(5\pi/3) = 1 \\ y = r\sin\theta &= 2\sin(5\pi/3) = -\sqrt{3}. \end{aligned}$$

[2 marks]

b) Convert $(-2, 2)$ from Cartesian to polar coordinates.

Solution.

$$r = \sqrt{4+4} = 2\sqrt{2}$$
$$\theta = \tan^{-1}(-1) + \pi = \frac{3\pi}{4}$$

[3 marks]

5. Determine whether the following limits exist. Where they exist, evaluate them.

a) $\lim_{x\to\pm\infty} \frac{2x^2+1}{x^2+x-2}$

Solution.

$$\lim_{x\to\pm\infty} \frac{2x^2+1}{x^2+x-2} = \lim_{x\to\pm\infty} \frac{2+\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}} = 2$$

[3 marks]

b) $\lim_{x\to 1} \frac{x^2+2x+3}{x^2+2x-3}$

Solution.

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^2+2x-3} = \lim_{x\to 1} \frac{(x-1)(x-2)}{(x+3)(x-1)} = \lim_{x\to 1} \frac{x-2}{x+3} = -\frac{1}{4}$$

Alternatively, since $x^2-3x+2 = 0 = x^2+2x-3$ and $x = 1$ we can apply l'Hopital's rule and obtain $\lim_{x\to 1} \frac{x^2-3x+2}{x^2+2x-3} = \lim_{x\to 1} \frac{2x-3}{2x+2} = -\frac{1}{4}$ [3 marks]

6. Differentiate the following functions. In part a), also find the tangent line through the point $(1, 1)$.

a) $f(x) = 2x^3 + 3x - 4$

Solution. a) $f'(x) = 6x^2 + 3$. The tangent line is $y - 1 = f'(1)(x - 1)$, that is, $y - 1 = 9(x - 1)$, or $y = 9x - 8$.

[4 marks]

b) $f(x) = \tan(2x)$

Solution Using the Chain Rule with $u = 2x$ and $f(x) = \tan u$, we have $f'(x) = 2\sec^2(2x)$.

[3 marks]

c) $f(x) = \frac{\sin x}{x^2}$

Solution Using the Product Rule, since $f(x) = x^{-2}\sin x$,

$$f'(x) = -2x^{-3}\sin x + x^{-2}\cos x = \frac{-2\sin x + x\cos x}{x^3}.$$

[4 marks]

7.

a) Find the Maclaurin series of $f(x) = \ln(1-x)$

Solution

$$f'(x) = -(1-x)^{-1},\ \ f''(x) = -(1-x)^{-2},\ \ f^{(3)}(x) = -2(1-x)^{-3}, \cdots f^{(n)}(x) = -n!(1-x)^{-n-1}$$

So

$$f(0) = \ln 1 = 0, \quad f'(0) = -1, \quad \frac{f''(0)}{2!} = -\frac{1}{2} \cdots \frac{f^{(n)}(0)}{n!} = -\frac{1}{n}.$$

So the Maclaurin Series of f is

$$-x - \frac{1}{2}x^2 \cdots - \frac{1}{n}x^n \cdots = -\sum_{n=1}^{\infty} \frac{1}{n}x^n.$$

[4 marks]

b) Hence, or otherwise, find the Maclaurin series of $g(x) = \ln(1-x^2)$

Solution We obtain the Maclaurin series for g simply by substituting x^2 for x in the Maclaurin series for f. So the Maclaurin series for g is

$$-x^2 - \frac{1}{2}x^4 \cdots - \frac{1}{n}x^{2n} \cdots = -\sum_{n=1}^{\infty} \frac{1}{n}x^{2n}.$$

[2 marks]