Chapter 1

Functions and Graphs

1.1 Numbers (1.2.1, 1.2.4)

The most fundamental type of number are those we use to count with: $0, 1, 2, \ldots$. These are called the *natural numbers*: the *set* of all natural numbers is denoted $\mathbb{N}$.

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Next we encounter the whole numbers or *integers*: the set of all integers is denoted $\mathbb{Z}$.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Clearly every natural number is an integer: that is $\mathbb{N} \subset \mathbb{Z}$.

Third, there are the fractions or *rational numbers*: the set of all rational numbers is denoted $\mathbb{Q}$. The rational numbers are those which can be written in the form p/q, where p and q are integers and $q \neq 0$. In set notation,

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}.$$

Since any integer n can be written as $n/1$, every integer is a rational number: that is $\mathbb{Z} \subseteq \mathbb{Q}$.

Finally there are the *real numbers*: all numbers which can be written with a decimal expansion. The set of all real numbers is denoted $\mathbb{R}$. Not every real number is rational: for example $\sqrt{2}$ and π can't be written in the form p/q. Such real numbers are called *irrational*.

Thus we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Later on we'll come across the *Complex numbers* $\mathbb{C}$.

Interval notation

Interval notation is a very convenient way of denoting sets of real numbers. If a and b are real numbers with $a \leq b$, we write $[a, b]$ for the set of all real numbers x with $a \leq x \leq b$. That is,

$$[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}.$$

Notice that this is really a collection of *real* numbers: thus $[1, 4]$ does not just contain the numbers $1, 2, 3, 4$, but everything between 1 and 4 (for example, π).

Similarly we use the notation (a, b) for the same set excluding the endpoints:

$$(a, b) = \{x \in \mathbb{R}: a < x < b\}.$$

We can mix square and round brackets:

$$\begin{aligned} [a, b) &= \{x \in \mathbb{R}: a \leq x < b\}. \\ (a, b] &= \{x \in \mathbb{R}: a < x \leq b\}. \end{aligned}$$

When we don't want an upper or lower limit, we can use the symbol ∞:

$$\begin{aligned} [a, \infty) &= \{x \in \mathbb{R}: a \leq x\} \\ (-\infty, b) &= \{x \in \mathbb{R}: x < b\} \end{aligned}$$

You should never put a square bracket next to ∞ or $-\infty$: ∞ is a convenient symbol, but it is *not* a real number.

1.2 Functions, Domain and Range (2.1, 2.2)

We often write expressions like $y = f(x)$. Here f is a *function*: we regard f as a machine, which, when we feed it a real number x, either spits out another real number $f(x)$ or tells us it doesn't like x. For example, if $f(x) = 1/x$, then if we feed f any real number $x \neq 0$, spits out the real number $1/x$: if we accidentally feed it $x = 0$, it complains (remember ∞ is not a real number).

Since we don't want our machine to complain, we have to be careful only to feed it allowable numbers.

The *Maximal Domain* of f is the set of all inputs x which don't make the machine complain (so $f(x)$ is a real number). Thus the maximal domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$. Sometimes we want to restrict the choice of inputs: a *domain* of f is any set of allowed inputs x: thus $[2, 5]$ is a domain of $f(x) = 1/x$, but $[-2, 2]$ is not (it contains 0, which is disallowed).

The *Range* of f is the set of possible output values y.

The *zeros* of f are all the possible input values x such that the output $f(x) = 0$. Also called *roots*.

1.3 Polynomials (2.4)

Polynomials are a very simple type of function:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

The *degree* of the polynomial is the largest power of x that appears.

1. Degree 0: constants $f(x) = c_0$.
2. Degree 1: linear functions $f(x) = c_0 + c_1 x$.
3. Degree 2: quadratics $f(x) = c_0 + c_1 x + c_2 x^2$.
4. Degree 3: cubics $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$.

Examples $f(x) = x^2 - 4$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $[-4, \infty)$. Two zeros, ± 2.

$f(x) = x^3 - 3x$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $\mathbb{R}$. Three zeros, 0 and $\pm\sqrt{3}$.

The maximal domain of a polynomial is always $\mathbb{R}$. Polynomials are also *continuous* (you can draw the graph without taking your pen off the paper) and *smooth* (there are no sharp corners in the graph).

1.4 Rational functions (2.5)

A rational function is one which can be written in the form

$$f(x) = \frac{g(x)}{h(x)},$$

where $g(x)$ and $h(x)$ are polynomials.

Example

$$\frac{x^3 - 3x^2 + 5}{2x^4 + x - 3}.$$

Unlike polynomials, the maximal domain of a rational function may not be $\mathbb{R}$: they explode whenever $h(x) = 0$. The zeros of a rational function are exactly the points where $g(x) = 0$.

Examples $f(x) = 1/x$. Draw graph. The maximal domain is $(-\infty, 0) \cup (0, \infty)$. The range is $(-\infty, 0) \cup (0, \infty)$. The line $x = 0$ is a *vertical asymptote*. f is not continuous: it jumps at $x = 0$. f has no zeros.

$f(x) = (x + 2)/(x - 1)^2$. Don't try to draw graph. The maximal domain is $(-\infty, 1) \cup (1, \infty)$. f has one zero, at $x = -2$.

1.5 Modulus (1.2.4)

The *modulus* $|x|$ of a real number x is just its size: thus $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

Examples $f(x) = |x|$. Draw graph. The maximal domain in $\mathbb{R}$. The range is $[0, \infty)$. There is one zero, at $x = 0$. f is continuous, but not smooth (there is a sharp corner at $x = 0$).

$f(x) = |x^2 - 4|$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $[0, \infty)$. There are two zeros, at $x = \pm 2$. f is continuous, but not smooth.

$f(x) = |x^2 + 1|$ is just the same as $f(x) = x^2 + 1$.

1.6 Even and Odd Functions (2.2.4)

An even function $f(x)$ is one for which $f(-x) = f(x)$ for all values of x (in the maximal domain). Thus the graph to the left of the y-axis can be obtained from the graph to the right by reflecting in the y-axis.

Examples are x^2, $|x|$, $x^4 + 2x^2 + 3$, any polynomial with only even powers.

An odd function $f(x)$ is one for which $f(-x) = -f(x)$ for all values of x (in the maximal domain). Thus the graph to the left of the y-axis can be obtained from the graph to the right by rotating about the origin.

Examples are x, $1/x$, $x^3 - 3x$, any polynomial with only odd powers.

Unlike numbers, most functions are neither even nor odd. Example $f(x) = x - 3$. Any polynomial with both even and odd powers is neither even nor odd.

To decide whether a function $f(x)$ is even, odd, or neither, work out $f(-x)$ and decide whether it is equal to $f(x)$, to $-f(x)$, or to neither of these.

Examples: $f(x) = \frac{x}{x+2}$, $f(x) = \sin(x^3)$, $f(x) = \sin(|x|)$, $f(x) = \frac{\sin(x)}{x}$. Note last is not defined at $x = 0$.

1.7 Increasing and Decreasing functions (2.2.1)

$f(x)$ is increasing on an interval $[a, b]$ if $f(x_1) \leq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$. Intuitively, the graph slopes upwards in $[a, b]$, but may have flat bits. It is strictly increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. (There are no flat bits).

$f(x)$ is decreasing on $[a, b]$ if $f(x_1) \geq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$, and strictly decreasing if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Example $f(x) = x^2 - 3$ is strictly increasing on $[0, \infty)$ (and indeed on $[3, 7]$), and strictly decreasing on $(-\infty, 0]$ (and indeed on $[-\pi, -\sqrt{2}]$.) It is neither increasing nor decreasing on $[-1, 1]$.