Chapter 1

Functions and Graphs

1.1 Numbers (1.2.1, 1.2.4)

The most fundamental type of number are those we use to count with: $0, 1, 2, \ldots$. These are called the *natural numbers*: the *set* of all natural numbers is denoted $\mathbb{N}$.

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Next we encounter the whole numbers or *integers*: the set of all integers is denoted $\mathbb{Z}$.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Clearly every natural number is an integer: that is $\mathbb{N} \subset \mathbb{Z}$.

Third, there are the fractions or *rational numbers*: the set of all rational numbers is denoted $\mathbb{Q}$. The rational numbers are those which can be written in the form p/q, where p and q are integers and $q \neq 0$. In set notation,

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}.$$

Since any integer n can be written as $n/1$, every integer is a rational number: that is $\mathbb{Z} \subseteq \mathbb{Q}$.

Finally there are the *real numbers*: all numbers which can be written with a decimal expansion. The set of all real numbers is denoted $\mathbb{R}$. Not every real number is rational: for example $\sqrt{2}$ and π can't be written in the form p/q. Such real numbers are called *irrational*.

Thus we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Later on we'll come across the *Complex numbers* $\mathbb{C}$.

Interval notation

Interval notation is a very convenient way of denoting sets of real numbers. If a and b are real numbers with $a \le b$, we write $[a, b]$ for the set of all real numbers x with $a \le x \le b$. That is,

$$[a, b] = \{x \in \mathbb{R}\colon a \le x \le b\}.$$

Notice that this is really a collection of *real* numbers: thus $[1, 4]$ does not just contain the numbers $1, 2, 3, 4$, but everything between 1 and 4 (for example, π).

Similarly we use the notation (a, b) for the same set excluding the endpoints:

$$(a, b) = \{x \in \mathbb{R}\colon a < x < b\}.$$

We can mix square and round brackets:

$$\begin{aligned}
[a, b) &= \{x \in \mathbb{R}\colon a \le x < b\}. \\
(a, b] &= \{x \in \mathbb{R}\colon a < x \le b\}.
\end{aligned}$$

When we don't want an upper or lower limit, we can use the symbol ∞:

$$\begin{aligned}
[a, \infty) &= \{x \in \mathbb{R}\colon a \le x\} \\
(-\infty, b) &= \{x \in \mathbb{R}\colon x < b\}
\end{aligned}$$

You should never put a square bracket next to ∞ or $-\infty$: ∞ is a convenient symbol, but it is *not* a real number.

1.2 Functions, Domain and Range (2.1, 2.2)

We often write expressions like $y = f(x)$. Here f is a *function*: we regard f as a machine, which, when we feed it a real number x, either spits out another real number $f(x)$ or tells us it doesn't like x. For example, if $f(x) = 1/x$, then if we feed f any real number $x \ne 0$, spits out the real number $1/x$: if we accidentally feed it $x = 0$, it complains (remember ∞ is not a real number).

Since we don't want our machine to complain, we have to be careful only to feed it allowable numbers.

The *Maximal Domain* of f is the set of all inputs x which don't make the machine complain (so $f(x)$ is a real number). Thus the maximal domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$. Sometimes we want to restrict the choice of inputs: a *domain* of f is any set of allowed inputs x: thus $[2,5]$ is a domain of $f(x) = 1/x$, but $[-2,2]$ is not (it contains 0, which is disallowed).

The *Range* of f is the set of possible output values y.

The *zeros* of f are all the possible input values x such that the output $f(x) = 0$. Also called *roots*.

1.3 Polynomials (2.4)

Polynomials are a very simple type of function:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

The *degree* of the polynomial is the largest power of x that appears.

1. Degree 0: constants $f(x) = c_0$.
2. Degree 1: linear functions $f(x) = c_0 + c_1 x$.
3. Degree 2: quadratics $f(x) = c_0 + c_1 x + c_2 x^2$.
4. Degree 3: cubics $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$.

Examples $f(x) = x^2 - 4$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $[-4, \infty)$. Two zeros, ± 2.

$f(x) = x^3 - 3x$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $\mathbb{R}$. Three zeros, 0 and $\pm\sqrt{3}$.

The maximal domain of a polynomial is always $\mathbb{R}$. Polynomials are also *continuous* (you can draw the graph without taking your pen off the paper) and *smooth* (there are no sharp corners in the graph).

1.4 Rational functions (2.5)

A rational function is one which can be written in the form

$$f(x) = \frac{g(x)}{h(x)},$$

where $g(x)$ and $h(x)$ are polynomials.

Example

$$\frac{x^3 - 3x^2 + 5}{2x^4 + x - 3}.$$

Unlike polynomials, the maximal domain of a rational function may not be $\mathbb{R}$: they explode whenever $h(x) = 0$. The zeros of a rational function are exactly the points where $g(x) = 0$.

Examples $f(x) = 1/x$. Draw graph. The maximal domain is $(-\infty, 0) \cup (0, \infty)$. The range is $(-\infty, 0) \cup (0, \infty)$. The line $x = 0$ is a *vertical asymptote.* f is not continuous: it jumps at $x = 0$. f has no zeros.

$f(x) = (x + 2)/(x - 1)^2$. Don't try to draw graph. The maximal domain is $(-\infty, 1) \cup (1, \infty)$. f has one zero, at $x = -2$.

1.5 Modulus (1.2.4)

The *modulus* $|x|$ of a real number x is just its size: thus $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

Examples $f(x) = |x|$. Draw graph. The maximal domain in $\mathbb{R}$. The range is $[0, \infty)$. There is one zero, at $x = 0$. f is continuous, but not smooth (there is a sharp corner at $x = 0$).

$f(x) = |x^2 - 4|$. Draw graph. The maximal domain is $\mathbb{R}$. The range is $[0, \infty)$. There are two zeros, at $x = \pm 2$. f is continuous, but not smooth.

$f(x) = |x^2 + 1|$ is just the same as $f(x) = x^2 + 1$.

1.6 Even and Odd Functions (2.2.4)

An even function $f(x)$ is one for which $f(-x) = f(x)$ for all values of x (in the maximal domain). Thus the graph to the left of the y-axis can be obtained from the graph to the right by reflecting in the y-axis.

Examples are x^2, $|x|$, $x^4 + 2x^2 + 3$, any polynomial with only even powers.

An odd function $f(x)$ is one for which $f(-x) = -f(x)$ for all values of x (in the maximal domain). Thus the graph to the left of the y-axis can be obtained from the graph to the right by rotating about the origin.

Examples are x, $1/x$, $x^3 - 3x$, any polynomial with only odd powers.

Unlike numbers, most functions are neither even nor odd. Example $f(x) = x - 3$. Any polynomial with both even and odd powers is neither even nor odd.

To decide whether a function $f(x)$ is even, odd, or neither, work out $f(-x)$ and decide whether it is equal to $f(x)$, to $-f(x)$, or to neither of these.

Examples: $f(x) = \frac{x}{x+2}$, $f(x) = \sin(x^3)$, $f(x) = \sin(|x|)$, $f(x) = \frac{\sin(x)}{x}$. Note last is not defined at $x = 0$.

1.7 Increasing and Decreasing functions (2.2.1)

$f(x)$ is increasing on an interval $[a, b]$ if $f(x_1) \leq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$. Intuitively, the graph slopes upwards in $[a, b]$, but may have flat bits. It is strictly increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. (There are no flat bits).

$f(x)$ is decreasing on $[a, b]$ if $f(x_1) \geq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$, and strictly decreasing if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Example $f(x) = x^2 - 3$ is strictly increasing on $[0, \infty)$ (and indeed on $[3, 7]$), and strictly decreasing on $(-\infty, 0]$ (and indeed on $[-\pi, -\sqrt{2}]$.) It is neither increasing nor decreasing on $[-1, 1]$.

1.8 Inverse functions (2.2.2)

Suppose f is a function: that is, if we input a real number x, it outputs a real number y. The *inverse function* f^{-1} is the function which takes the output of f and tells us what the input was. Thus if $y = f(x)$ then $x = f^{-1}(y)$.

Example $f(x) = x + 3$. Then $f^{-1}(x) = x - 3$. (If $y = x + 3$, then $x = y - 3$, so $f^{-1}(y) = y - 3$. However a function doesn't care what letter I use to define it, so we can also write $f^{-1}(x) = x - 3$.) Draw graphs.

Thus determining the inverse function boils down to solving the equation $y = f(x)$ for x in terms of y.

Example $f(x) = x/(x - 2)$. To find the inverse function, write $y = x/(x - 2)$ and solve for x. $y(x - 2) = x$, $yx - x = 2y$, $x(y - 1) = 2y$, $x = 2y/(y - 1)$. Thus $f^{-1}(y) = 2y/(y - 1)$, or $f^{-1}(x) = 2x/(x - 1)$. Show graphs.

Notice the *reflection rule*: since finding the inverse function is just interchanging the roles of x and y, the graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected in the line $y = x$.

Problem: not every function has an inverse. Consider for example $f(x) = x^2$. Should we say that $f^{-1}(4) = 2$, or -2? We can't decide. Can see this problem on the graphs: draw graph of x^2 and reflection in $y = x$. The problem is that $f(x) = x^2$ is *two to one*: there are two values of x which give rise to each value $f(x)$.

We say that $f(x)$ is *one to one* (or $1 - 1$) if different values of x always give different values of $f(x)$: that is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

One to one functions $f(x)$ always have inverses: the maximal domain of $f^{-1}(x)$ is the range of $f(x)$, so may not be the same as the maxiaml domain of $f(x)$.

Examples $f(x) = x^3$ is $1 - 1$. The maximal domain and the range of $f(x)$ are $\mathbb{R}$, and the same is true of $f^{-1}(x) = \sqrt[3]{x}$.

$f(x) = e^x$ is $1 - 1$. $f(x)$ has maximal domain $\mathbb{R}$, and range $(0, \infty)$. The inverse $f^{-1}(x) = \ln(x)$ has maximal domain $(0, \infty)$, and range $\mathbb{R}$. (Pretend we know about these functions for now.)

If $f(x)$ is not $1-1$, and we want to talk about its inverse, we have to restrict the domain of $f(x)$ to one where it is $1 - 1$.

Example $f(x) = x^2$ is not $1 - 1$ on its maximal domain, but it is $1 - 1$ on the domain $[0, \infty)$. If we restrict to this domain, then $f(x)$ has an inverse $f^{-1}(x) = +\sqrt{x}$. Draw graphs. (Could just as well have chosen other domain).

Note: if $f(x)$ is continuous, then it is $1 - 1$ on an interval precisely when it is either strictly increasing or strictly decreasing there. Pictures.

1.9 Trigonometric Functions (2.6)

Draw a circle of radius 1, and pick a point $P = (x, y)$ on the circle, at angle θ to the horizontal.

Then $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = y/x (= \sin\theta/\cos\theta)$.

Notice that $-1 \leq \sin\theta, \cos\theta \leq 1$ for any value of θ: i.e. the *range* of sin and cos is $[-1, 1]$. On the other hand, $\tan\theta$ can take any real value: the *range* of tan is $\mathbb{R}$.

By pythagoras's theorem, $\sin^2\theta + \cos^2\theta = 1$.

You should *always* express angles in radians rather than degrees: there are very good reasons for this, which will become clear during the course. Remember that a full revolution (360 degrees) is 2π radians, so (give values of 180, 90, 60, 30 degrees).

Special angles: (Use 30-60-90 and 45-45-90 rules)

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	

Graphs of the trigonometric functions

Draw graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$. Note that cos is even and sin and tan are odd.

A function $f(x)$ is *periodic with period* T or T*-periodic* if $f(x+T) = f(x)$ for all x.

Thus sin and cos are 2π-periodic, tan is π-periodic.

Also define $\cot\theta = 1/\tan\theta = \cos\theta/\sin\theta$, $\operatorname{cosec}\theta = 1/\sin\theta$ and $\sec\theta = 1/\cos\theta$.

Draw graph of $\cot\theta$: odd and π-periodic.

Trigonometric identities

Hand out trigonometric identities and talk about them.

We can derive other identities from these.

Example To get an expression for $\cos(3\theta)$ in terms of $\cos\theta$, write $\cos(3\theta) = \cos(2\theta+\theta)$, and observe

$$\begin{aligned}
\cos(2\theta+\theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \quad (4)\\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \quad (10, 11)\\
&= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\
&= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \quad (1)\\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\\
&= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

Check: Try $\theta = 0$. Then $\cos 3\theta = \cos 0 = 1$, and $\cos\theta = 1$, so the right hand side is $4(1)^3 - 3 = 4 - 3 = 1$. So this checks. Try $\theta = \pi/3$. Then $\cos 3\theta = \cos\pi = -1$, while $\cos\theta = \cos\pi/3 = 1/2$. Thus the right hand side is $4(1/2)^3 - 3/2 = 1/2 - 3/2 = -1$. So this checks.

Inverse trigonometric functions

The trigonometric functions are periodic, and so are "∞ to 1". In order to consider their inverse functions, we have to restrict their domain to a *principal domain*, just as we did for $f(x) = x^2$.

The principal domain of $y = \sin x$ (draw graph) is $[-\pi/2, \pi/2]$. Thus the *principal value* of $\sin^{-1} x$ lies in $[-\pi/2, \pi/2]$. (There are infinitely many angles whose sine is x: we pick the one which lies between $-\pi/2$ and $\pi/2$.) Draw graph. Maximal domain $[-1,1]$, range $[-\pi/2,\pi/2]$.

The principal domain of $y=\cos x$ (draw graph) is $[0,\pi]$. Thus the *principal value* of $\cos^{-1} x$ lies in $[0,\pi]$. Draw graph. Maximal domain $[-1,1]$, range $[0,\pi]$.

The principal domain of $y=\tan x$ (draw graph) is $[-\pi/2,\pi/2]$. Thus the *principal value* of $\tan^{-1} x$ lies in $[-\pi/2,\pi/2]$. Draw graph. Maximal domain $\mathbb{R}$, range $[-\pi/2,\pi/2]$.

NB On your calculator, $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ may be called $\arcsin x$, $\arccos x$, and $\arctan x$. Your calculator should automatically give the principal values of these functions. Make sure it's set on radians.

Trigonometric Equations

Consider solving the equation $\sin\theta = 1/2$ for θ. One solution is $\theta = \sin^{-1}(1/2) = \pi/6$. However there are infinitely many other solutions (draw graph). These are $\pi/6 + 2n\pi$, where n is any integer, and $(\pi-\pi/6)+2n\pi$, where n is any integer. Thus the *general solution* of $\sin\theta = 1/2$ is

$$\theta = \begin{cases} \pi/6 + 2n\pi & n\in\mathbb{Z} \\ (\pi-\pi/6)+2n\pi & n\in\mathbb{Z}. \end{cases}$$

We can rewrite the equation as $\sin\theta = \sin\pi/6$. By the same argument, the *general solution* of the equation $\sin\theta=\sin\alpha$ is

$$\theta = \begin{cases} \alpha + 2n\pi & n\in\mathbb{Z} \\ (\pi-\alpha)+2n\pi & n\in\mathbb{Z}. \end{cases}$$

Example Find the general solution of $\sin\theta = 1/\sqrt{2}$. Write $1/\sqrt{2} = \sin\alpha$: here $\alpha + \sin^{-1}(1/\sqrt{2}) = \pi/4$. Thus the general solution is

$$\theta = \begin{cases} \pi/4 + 2n\pi & n\in\mathbb{Z} \\ (\pi-\pi/4)+2n\pi & n\in\mathbb{Z}. \end{cases}$$

Analogously, the general solution of the equation $\cos\theta=\cos\alpha$ is

$$\theta = \pm\alpha + 2n\pi \qquad n\in\mathbb{Z},$$

and the general solution of the equation $\tan\theta = \tan\alpha$ is

$$\theta = \alpha + n\pi \qquad n\in\mathbb{Z}.$$

Example Find the general solution of $\tan\theta = 3$. Write $3 = \tan\alpha$: thus $\alpha = \tan^{-1}(3) = 1.2490 = 0.3976\pi$. Thus the general solution is $\theta = 0.3976\pi + n\pi$, or $\theta = (0.3976 + n)\pi$. That is, the values of θ with $\tan\theta = 3$ are 0.3976π, 1.3976π, 2.3976π, ... and -0.6024π, -1.6024π, -2.6024π, etc.

A second type of trigonometric equation can be solved by a trick. These are equations of the form $a\cos\theta + b\sin\theta = c$, where a, b, and c are fixed. To solve this, consider a right-angled triangle with sides a and b, and hypoteneuse $R = \sqrt{a^2 + b^2}$: let ϕ be the angle between a and R (so $\phi = \tan^{-1}(b/a)$). Then $a = R\cos\phi$ and $b = R\sin\phi$. Thus

$$a\cos\theta + b\sin\theta = R\cos\phi\cos\theta + R\sin\phi\sin\theta = R\cos(\theta - \phi).$$

Our equation therefore becomes

$$\cos(\theta - \phi) = c/R,$$

which we can solve in the usual way. Writing $c/R = \cos\alpha$, we have

$$\cos(\theta - \phi) = \cos\alpha,$$

which has general solution

$$\theta - \phi = \pm\alpha + 2n\pi,$$

or

$$\theta = \pm\alpha + 2n\pi + \phi.$$

Since $R = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$, we have $\alpha = \cos^{-1}(c/\sqrt{a^2 + b^2}$, so we can write the general solution of $a\cos\theta + b\sin\theta = c$ as

$$\theta = \pm\cos^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right) + 2n\pi + \tan^{-1}\frac{b}{a}.$$

However, rather than remember this formula, it is better to work through the steps for each example.

Example Find the general solution of the equation $\cos\theta + 2\sin\theta = 1$.

Consider the triangle with sides 1, 2, and $\sqrt{5}$, and let ϕ be the angle $\tan^{-1} 2 = 1.1071$. Then $1 = \sqrt{5}\cos\phi$ and $2 = \sqrt{5}\sin\phi$. Hence $\cos\theta + 2\sin\theta = \sqrt{5}\cos\theta\cos\phi + \sqrt{5}\sin\theta\sin\phi = \sqrt{5}\cos(\theta - \phi)$.

Solving $\cos\theta + 2\sin\theta = 1$ is the same as solving $\sqrt{5}\cos(\theta - \phi) = 1$, or $\cos(\theta - \phi) = 1/\sqrt{5}$. Now $\cos^{-1}(1/\sqrt{5}) = 1.1071$, so the general solution is

$$\theta - \phi = \pm 1.1071 + 2n\pi,$$

or

$$\theta = \phi \pm 1.1071 + 2n\pi = 1.1071 \pm 1.1071 + 2n\pi$$

so $\theta = 2n\pi$ or $\theta = 2.2142 + 2n\pi$.

The first type of solution is easy to check: if $\theta = 2n\pi$ then $\sin\theta = 0$ and $\cos\theta = 1$, so $\cos\theta + 2\sin\theta = 1$. For the second type, we can check that $\cos(2.2142) = -0.6$ and $\sin(2.2142) = 0.8$, so $\cos(2.2142) + 2\sin(2.2142) = 1$.

1.10 Polar Coordinates (2.6.6)

Sometimes, instead of describing a point P by its *Cartesian coordinates* (x, y) (the horizontal and vertical distances from the origin), it's convenient to represent it by its distance r and angle θ from the origin. Draw picture.

To convert from Cartesian to polar coordinates, use the formulae

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = y/x,$$

and to convert from polar to Cartesian coordinates, use the formulae

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Examples Let P be the point with Cartesian coordinates $(2, 1)$. To find its polar coordinates: $r = \sqrt{2^2 + 1^2} = \sqrt{5}$, and $\tan\theta = 1/2$ so $\theta = \tan^{-1}(1/2) = 0.4636 (= 0.1476\pi)$. So the polar coordinates of P are $(r, \theta) = (\sqrt{5}, 0.1476\pi)$.

Let P be the point with polar coordinates $(2, \pi/3)$. To find its Cartesian coordinates: $x = 2\cos\pi/3 = 2(1/2) = 1$ and $y = 2\sin\pi/3 = 2(\sqrt{3}/2) = \sqrt{3}$.

Note:

a) When P is the origin, θ is not defined.

b) Beware when using the formula $\tan\theta = y/x$ to calculate θ: when you calculate $\tan^{-1}(y/x)$ on your calculator, it will return the *principal value* of $\tan^{-1}(y/x)$, which lies between $-\pi/2$ and $\pi/2$ (i.e. $x > 0$). When determining θ, you have to look at the sign of x: if $x > 0$ then $\theta = \tan^{-1}(y/x)$; if $x < 0$ then $\theta = \tan^{-1}(y/x) + \pi$; if $x = 0$ then $\theta = \pi/2$ or $-\pi/2$ depending on whether $y > 0$ or $y < 0$. For example, let P have Cartesian coordinates $(-2, -2)$. Then $r = \sqrt{8}$. If we work out $\tan^{-1}(-2/-2) = \tan^{-1}(1)$, we get $\pi/4$, which is clearly wrong. Since $x < 0$, we have to add π to this to get θ: $\theta = 5\pi/4$.

1.11 Limits (7.8, 7.9)

We look at the behaviour of $f(x)$ as x approaches a certain value. Let's start with some intuitive examples:

Examples

a) $f(x) = x^2$. Clearly when x is very close to 2, $f(x)$ is very close to 4. We say $f(x) \to 4$ as $x \to 2$, or $\lim_{x\to 2} f(x) = 4$. (Boring: this is because x^2 is *continuous* at $x = 2$).

b) $f(x) = 1/x$. When x is negative and very close to 0, $f(x)$ is a very large negative number. When x is postive and very close to 0, $f(x)$ is a very large positive number. We say $\lim_{x\to 0-} f(x) = -\infty$ and $\lim_{x\to 0+} f(x) = \infty$. Althoug we say this, $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+}$ do not exist, strictly speaking, since the limit always has to be a number.

c) Now consider the *Heaviside step function*

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

We have $\lim_{x\to 0-} f(x) = 0$ and $\lim x \to 0{+}f(x) = 1$. Thus $\lim_{x\to 0} f(x)$ doesn't exist. However, if we look close to $x = 2$ then clearly $\lim_{x\to 2} f(x) = 1$.

d) Notice that the limit says nothing at all about $f(a)$ itself: if we defined a function by

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 99 & \text{if } x = 2, \end{cases}$$

Then we still have $\lim_{x\to 2} f(x) = 4$, even though $f(2) = 99$.

This motivates the definition of continuity:
The function $f(x)$ is *continuous* at $x = a$ if

a) a is in the maximal domain of $f(x)$.

b) $\lim_{x\to a} f(x) = f(a)$.

$f(x)$ is *continuous* if it is continuous at all values of x.

Examples $f(x) = x^2$ is continuous. The Heaviside step function is continuous everywhere except at $x = 0$. $f(x) = 1/x$ is continuous everywhere except at $x = 0$.

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 99 & \text{if } x = 2 \end{cases}$$

is not continuous at $x = 2$, since $\lim_{x\to 2} f(x)$ isn't equal to $f(2)$.

Some more examples: Rational functions

Example 1: $f(x) = (x^2 + 3)/(x - 2)$ as $x \to 1$. Clearly when x is very close to 1, $f(x)$ is very close to $f(1) = -4$. Hence $\lim_{x\to 1} f(x) = -4$, and $f(x)$ is continuous at $x = 1$.

Example 2: $f(x) = (x^2 + 3)/(x - 2)$ as $x \to 2$. When x is very close to 2, then x^2+3 is very close to 7. However as x gets closer and closer to 2 from above, $x-2$ becomes a smaller and smaller positive number. Hence $\lim_{x\to 2+} f(x) = \infty$. When x tends to 2 from below, $x-2$ becomes a smaller and smaller negative number: hence $\lim_{x\to 2-} f(x) = -\infty$.

Example 3: You should always try to simplify $f(x)$ before calculating the limit.

Consider $f(x) = (x^2-1)/(x-1)$ as $x \to 1$. Simplifies to $f(x) = x+1$ *provided that* $x \neq 1$. Hence $\lim_{x\to 1} f(x) = 2$. Note, however, that $f(x)$ is not continuous at $x = 1$, since $f(1) = (1^2 - 1)/(1 - 1)$ is not defined, i.e. 1 isn't in the maximal domain of $f(x)$.

A very important example: sinc$x$

Consider the function $f(x) = \sin x/x$ as $x \to 0$. Show graph. Appears that $\lim_{x\to 0} \sin x/x = 1$. Give geometric interpretation.

Notice that $f(x)$ is *not* continuous at $x = 0$, since $f(0) = \sin 0/0$ is not defined. However, we can define a new function

$$\mathrm{sinc}x = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then sinc$x$ is continuous at $x = 0$.

Another type of example can be solved using a trick. Consider $f(x) = \sin 2x/x$. Then we can write $f(x) = 2\sin 2x/2x$. As $x \to 0$, $2x \to 0$ also, so $\sin 2x/2x \to 1$ as $x \to 0$. Hence $f(x) \to 2$ as $x \to 0$.

Limits as $x \to \pm\infty$

Sometimes these limits exist: $\lim_{x\to\infty} 1/x = 0$, $\lim_{x\to-\infty} 1/x = 0$. Sometimes they don't: $\lim_{x\to\infty} \sin x$ doesn't exist.

If f is a non-constant polynomial, then $\lim_{x\to+\infty} f(x)$ is equal to $+\infty$ or $-\infty$, depending on the sign of the highest degree term, and similarly for $\lim_{x\to-\infty} f(x)$. So strictly speaking, the limit does not exist, asince $\pm\infty$ are not numbers.

a) $f(x) = x^2 + 3x + 3 \sim x^2 \to +\infty$ as $x \to +\infty$ and as $x \to -\infty$.

b) $f(x) = 2x^3 - 3x^2 - 2 =\sim 2x^3 \to +\infty$ as $x \to +\infty$ and $\to -\infty$ as $x \to -\infty$.

For a rational function f, it depends on the degrees of the polynomial on the top and bottom.

a) If the degree of the numerator of f is greater than the degree of th denominator of f, then $\lim_{x\to+\infty} f(x) = +\infty$ or $-\infty$ depending on the signs of the highest degree terms in numerator and denominator, and similarly for $\lim_{x\to-\infty} f(x)$.

$$\begin{aligned} f(x) &= \frac{x^3+1}{3x^2-2x+1} \\ &\sim \frac{x^3}{3x^2} \\ &= \frac{x}{3}, \end{aligned}$$

so $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

b) If the degree of the numerator is less than the degree of the denominator, then the limit is 0:

$$\begin{aligned} f(x) &= \frac{x^3+1}{2x^4-x^2+2} \\ &\sim \frac{x^3}{2x^4} \\ &= \frac{1}{2x}, \end{aligned}$$

so $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

c) If the degrees are the same, the limit is a non-zero real number:

$$\begin{aligned} f(x) &= \frac{x^3+1}{2x^3-3x+2} \\ &\sim \frac{x^3}{2x^3} \\ &= \frac{1}{2}, \end{aligned}$$

so $f(x) \to 1/2$ as $x \to +\infty$ and as $x \to -\infty$.

The sandwich rule

Suppose that $g(x) \leq f(x) \leq h(x)$ for all large x, and that $g(x) \to 0$ and $h(x) \to 0$ as $x \to \infty$. Then $f(x) \to 0$ as $x \to \infty$.

Example Consider $f(x) = \sin x/x$ as $x \to \infty$. Since $\sin x$ always lies between -1 and 1, we have

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

for all $x > 0$, and hence $\sin x/x \to 0$ as $x \to \infty$.

Asymptotes

Recall that we talked about horizontal and vertical asymptotes earlier: for example $f(x) = 1/(x-1)$ has a vertical asymptote $x = 1$ and a horizontal asymptote $y = 0$. We can now define these terms:

The line $x = a$ is a *vertical asymptote* of $f(x)$ if $\lim_{x\to a-} f(x) = \pm\infty$ or $\lim_{x\to a+} f(x) = \pm\infty$ or both.

The line $y = b$ is a *horizontal asymptote* of $f(x)$ if $\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ or both.