Chapter 2

Differentiation (8.1–8.3, 9.5)

2.1 Rate of Change (8.2.1–5)

Recall that the equation of a straight line can be written as $y = mx + c$, where m is the *slope* or *gradient* of the line, and c is the *y-intercept* (i.e. the value of y when $x = 0$).

Example $y = 2x + 1$. Draw it. The slope 2 can also be looked on as the *rate of change* of y with respect to x: when x increases by 1, y increases by 2. For example, if x represents time in seconds, and y represents distance travelled in meters, then the rate of change of y with respect to x is the speed of travel.

If the relationship between y and x is more complicated, for example $y = x^2$, then the rate of change of y wrt x is different for different values of x.

Example What is the rate of change of y wrt x when $x = 1$? When $x = 1$, $y = 1$. If x increases by a small amount δ, then y increases to $(1+\delta)^2 = 1+2\delta+\delta^2$, in other words y increases by $2\delta + \delta^2$. Thus

$$\text{Rate of change} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{2\delta + \delta^2}{\delta} = 2 + \delta.$$

To find the instantaneous rate of change at $x = 1$, we let $\delta \to 0$, to obtain 2. Thus the car is travelling at 2 m/s at time 1.

In general, let $y = f(x)$. The rate of change of y with respect to x at $x = x_0$ is given by

$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{\delta \to 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta}.$$

Example Return to the example $y = f(x) = x^2$, and let x_0 be any value of x. Then

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{x_0} &= \lim_{\delta \to 0} \frac{f(x_0 + \delta) - f(x_0)}{\delta} \\
&= \lim_{\delta \to 0} \frac{(x_0 + \delta)^2 - x_0^2}{\delta} \\
&= \lim_{\delta \to 0} \frac{x_0^2 + 2x_0\delta + \delta^2 - x_0^2}{\delta} \\
&= \lim_{\delta \to 0} (2x_0 + \delta) \\
&= 2x_0.
\end{aligned}$$

Thus at time x_0, the speed of the car is $2x_0$. Equivalently, at time x the speed of the car is $2x$. We also write

$$\frac{dy}{dx} = 2x, \qquad y' = 2x, \qquad \frac{df}{dx} = 2x, \qquad \text{or } f'(x) = 2x.$$

The rate of change is called the *derivative of y wrt x*, or the *derivative of $f(x)$ wrt x*, or just the *derivative of $f(x)$*.

Geometrically $f'(x_0)$ is the slope of the tangent to $y = f(x)$ at $x = x_0$ (picture). Thus the equation of this tangent is $y = f'(x_0)x + c$, where c is the y-intercept. In order to work out c, we use the fact that the tangent passes through the point $(x_0, f(x_0))$. Putting $x = x_0$ and $y = f(x_0)$ in the equation we get $f(x_0) = f'(x_0)x_0 + c$, so $c = f(x_0) - f'(x_0)x_0$, and hence the equation of the tangent is

$$y = f'(x_0)x + f(x_0) - f'(x_0)x_0,$$

or

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Example Find the equation of the tangent to the curve $y = x^2$ at $x_0 = 3$.

When $x_0 = 3$ we have $f(x_0) = 9$, and $f'(x_0) = 2x_0 = 6$. Hence the equation of the tangent is

$$y = 9 + 6(x - 3)$$

or

$$y = 6x - 9.$$

2.2 Derivatives of common functions: rules of differentiation (8.3.1–7)

Recall that if $f(x) = x^2$, then $f'(x) = 2x$. We found this with our bare hands:

$$f'(x) = \lim_{\delta \to 0} \frac{(x+\delta)^2 - x^2}{\delta} = \lim_{\delta \to 0} 2x + \delta = 2x.$$

We can do the same thing for other common functions.

Example Let $f(x) = x^3$. Then

$$\begin{aligned} f'(x) &= \lim_{\delta \to 0} \frac{(x+\delta)^3 - x^3}{\delta} \\ &= \lim_{\delta \to 0} \frac{x^3 + 3x^2\delta + 3x\delta^2 + \delta^3 - x^3}{\delta} \\ &= \lim_{\delta \to 0} (3x^2 + 3x\delta + \delta^2) \\ &= 3x^2. \end{aligned}$$

Thus

$$\frac{d}{dx} x^3 = 3x^2.$$

To find the derivative of x^n for other values of n, we need to be able to work out $(x+\delta)^n$. To do this, we have the *binomial theorem*: to work out $(a+b)^n$, we don't have to work out

$$(a+b)(a+b)(a+b)\ldots(a+b),$$

we can use

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \cdots + \binom{n}{n-1} ab^{n-1} + b^n,$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Rather than work out the coefficients $\binom{n}{r}$ using this formula, we can use *Pascal's triangle*. Draw it. Thus, for example

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Example Expand $(1+2x)^5$ using the binomial theorem.

$$\begin{aligned}(1+2x)^5 &= 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5 \\ &= 1 + 5(2x) + 10(4x^2) + 10(8x^3) + 5(16x^4) + (32x^5) \\ &= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.\end{aligned}$$

We can use this to work out the derivative of x^n for any n. Let $f(x) = x^n$. Then

$$\begin{aligned}f'(x) &= \lim_{\delta\to 0} \frac{(x+\delta)^n - x^n}{\delta} \\ &= \lim_{\delta\to 0} \frac{x^n + nx^{n-1}\delta + \text{terms in } \delta^2,\ \delta^3 \text{ etc.} - x^n}{\delta} \\ &= \lim_{\delta\to 0}(nx^{n-1} + \text{terms in } \delta,\ \delta^2 \text{ etc.}) \\ &= nx^{n-1}.\end{aligned}$$

Thus

$$\frac{d}{dx}x^n = nx^{n-1}.$$

This gives $\frac{d}{dx}x^2 = 2x$ and $\frac{d}{dx}x^3 = 3x^2$ in agreement with our earlier calculations. We can also now calculate, for example

$$\frac{d}{dx}x^{57} = 57x^{56}.$$

Example Calculate the equation of the tangent to the graph $y = x^{28}$ at $x = 1$.

Write $y = f(x) = x^{28}$. We want to use the formula for the tangent at $x = x_0$:

$$y = f(x_0) + f'(x_0)(x - x_0),$$

so since $x_0 = 1$ the equation is

$$y = f(1) + f'(1)(x-1).$$

Now $f(1) = 1^{28} = 1$, and $f'(x) = 28x^{27}$, so $f'(1) = 28$. Hence the equation of the tangent is

$$y = 1 + 28(x - 1),$$

or

$$y = 28x - 27.$$

Derivative of $\sin x$ and $\cos x$

Let $f(x) = \sin x$. We can calculate $f'(x)$ using what trigonometric identity (16):

$$\begin{aligned} f'(x) &= \lim_{\delta \to 0} \frac{\sin(x + \delta) - \sin x}{\delta} \\ &= \lim_{\delta \to 0} 2 \frac{\cos\left(\frac{2x+\delta}{2}\right) \sin\left(\frac{\delta}{2}\right)}{\delta} \\ &= \lim_{\delta \to 0} \cos(x + \frac{\delta}{2}) \frac{\sin(\delta/2)}{(\delta/2)} \\ &= \cos x. \end{aligned}$$

Thus $\frac{d}{dx} \sin x = \cos x$.

Similarly $\frac{d}{dx} \cos x = -\sin x$ (exercise).

Example Find the equation of the tangent to the graph $y = \sin x$ at $x = 0$.

Write $f(x) = \sin x$ and $x_0 = 0$. We want to use our formula

$$y = f(x_0) + f'(x_0)(x - x_0)$$

for the equation of the tangent. We have $f(x_0) = \sin 0 = 0$ and $f'(x_0) = \cos 0 = 1$, so the equation is

$$y = 0 + 1(x - 0),$$

or $y = x$.

To find derivatives of other functions, we need some *rules of differentiation*

The constant multiplication rule

If k is a constant, then $\frac{d}{dx} kf(x) = kf'(x)$.

Examples

a) $\frac{d}{dx} 3x^2 = 3(2x) = 6x$.

b) $\frac{d}{dx}5x^4 = 20x^3$.

c) $\frac{d}{dx}2\sin x = 2\cos x$.

The sum rule

If u and v are functions of x, then $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$. Alternatively, $(u+v)' = u' + v'$.

Examples

a) $\frac{d}{dx}(x^3 + 2x + 1) = 3x^2 + 2$. Similarly, we can work out the derivative of any polynomial.

b) $\frac{d}{dx}(x^2 + 2\sin x - \cos x) = 2x + 2\cos x + \sin x$.

The product rule

If u and v are functions of x, then $(uv)' = uv' + u'v$.

Examples

a) Let $f(x) = x^2 \sin x$. We let $u = x^2$ and $v = \sin x$. Thus $u' = 2x$ and $v' = \cos x$. The product rule says that $f'(x) = x^2 \cos x + 2x \sin x$.

b) Let $f(x) = \cos^2 x = \cos x \cos x$. We let $u = v = \cos x$. Then $u' = v' = -\sin x$. The product rule says that $f'(x) = \cos x(-\sin x) + (-\sin x)\cos x = -2\sin x \cos x$. Note $f'(x) = -\sin(2x)$.

c) Let $f(x) = x^2 \sin x \cos x$. We let $u = x^2 \sin x$ and $v = \cos x$. Thus $u' = x^2 \cos x + 2x \sin x$ (part a)), and $v' = -\sin x$. The product rule says that

$$f'(x) = (x^2 \sin x)(-\sin x) + (x^2 \cos x + 2x \sin x)\cos x = x^2(\cos^2 x - \sin^2 x) + 2x \sin x \cos x.$$

(Note $f'(x) = x^2 \cos 2x + x \sin 2x$.)

The quotient rule

If u and v are functions of x, then

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

Examples

a) Let $f(x) = 1/x$. We let $u = 1$ and $v = x$, so $u' = 0$ and $v' = 1$. The quotient rule says that

$$f'(x) = \frac{x(0) - (1)(1)}{x^2} = -1/x^2.$$

b) Let $f(x) = \tan x = \frac{\sin x}{\cos x}$. We let $u = \sin x$ and $v = \cos x$. Thus $u' = \cos x$ and $v' = -\sin x$. Thus

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

c) Let $f(x) = 1/x^n$. We let $u = 1$ and $v = x^n$, so $u' = 0$ and $v' = nx^{n-1}$. The quotient rule says that

$$f'(x) = \frac{x^n(0) - (1)nx^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}}.$$

Written another way,

$$\frac{d}{dx}x^{-n} = -nx^{-n-1},$$

so we can see that

$$\frac{d}{dx}x^n = nx^{n-1}$$

whether n is positive or negative. In fact, we have $\frac{d}{dx}x^a = ax^{a-1}$ for *any* number a. Some examples:

d) Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = (1/2)x^{-1/2} = \frac{1}{2\sqrt{x}}$.

e) Let $f(x) = \frac{1}{\sqrt[3]{x}} = x^{-1/3}$. Then $f'(x) = -(1/3)x^{-4/3} = \frac{-1}{3x\sqrt[3]{x}}$.

The chain rule

Let $f(x) = g(h(x))$. Then $f'(x) = g'(h(x))h'(x)$.

Examples

a) Let $f(x) = (4x-1)^3$. Let $g(x) = x^3$ and $h(x) = 4x - 1$, so $f(x) = g(h(x))$. We have $g'(x) = 3x^2$ and $h'(x) = 4$. Thus

$$f'(x) = g'(h(x))h'(x) = 3(4x-1)^2 \cdot 4 = 12(4x-1)^2.$$

b) Let $f(x) = \sin(3x+2)$. Let $g(x) = \sin x$ and $h(x) = 3x+2$, so $f(x) = g(h(x))$. We have $g'(x) = \cos x$ and $h'(x) = 3$. Thus

$$f'(x) = g'(h(x))h'(x) = \cos(3x+2) \cdot 3 = 3\cos(3x+2).$$

More generally, $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$ and $\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$.

c) Let $f(x) = (\sin x + \cos 3x)^3$. Let $g(x) = x^3$ and $h(x) = \sin x + \cos 3x$, so $f(x) = g(h(x))$. We have $g'(x) = 3x^2$ and $h'(x) = \cos x - 3\sin 3x$. Thus

$$f'(x) = g'(h(x))h'(x) = 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x).$$

d) Let $f(x) = \tan((\sin x + \cos 3x)^3)$. Let $g(x) = \tan x$ and $h(x) = (\sin x + \cos 3x^3)$, so $f(x) = g(h(x))$. We have $g'(x) = \sec^2(x)$ and $h'(x) = 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x)$, so

$$f'(x) = g'(h(x))h'(x) = \sec^2((\sin x + \cos 3x)^3) \cdot 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x).$$

The Inverse Function Rule

Let $y = f^{-1}(x)$ (so $x = f(y)$). Then

$$\frac{dy}{dx} = \frac{1}{f'(y)}.$$

Examples

a) Let $y = \sqrt{x}$ (so $x = y^2$, and we have $f(y) = y^2$. Then

$$\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}.$$

Thus

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

This agrees with our earlier way of calculating this: $\frac{d}{dx}x^{1/2} = frac12x^{-1/2}$.

b) Let $y = \sin^{-1}(x)$ (so $x = \sin y$, and we have $f(y) = \sin y$. Then

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Thus

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

c) Similarly, it can be shown that

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}.$$

d) Let $y = \tan^{-1}(x)$ (so $x = \tan y$, and we have $f(y) = \tan y$. Then

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Thus

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

2.3 An application: the Newton-Raphson method (9.5.8)

This is a method for getting an approximate solution to the equation $f(x) = 0$ in cases where we can't get an exact solution. Suppose that, by drawing a graph of $f(x)$ we can see that there is a solution α (so $f(\alpha) = 0$). The aim is to get a good approximation to α. From the graph we can make an initial guess x_0 at α. The idea (draw picture) is that the place x_1 where the tangent to the graph at x_0 hits the x-axis is a better approximation than x_0.

The equation of the tangent is

$$y = f(x_0) + f'(x_0)(x - x_0),$$

which intersects the x-axis when $y = 0$, so

$$f(x_0) + f'(x_0)(x - x_0) = 0,$$

or

$$x - x_0 = \frac{-f(x_0)}{f'(x_0)}.$$

Thus the tangent hits the x-axis when

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Thus

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now we can take x_1 as our new guess for α, and use the same method to get a better guess

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

We can repeat this as many times as we like to get better and better guesses x_3, x_4, and so on. In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example Consider the equation $x = \cos x$. By drawing the graphs of x and $\cos x$, we can see that there is a solution somewhere between $x = 0$ and $x = \pi/2$. Let's take $x_0 = 1$ as our initial guess at the solution.

We need to write the equation in the form $f(x) = 0$, which we do by setting $f(x) = x - \cos x$. Then $f'(x) = 1 + \sin x$. Thus the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

becomes

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}.$$

So

$$x_1 = x_0 - \frac{x_0 - \cos x_0}{1 + \sin x_0} = 1 - \frac{1 - \cos(1)}{1 + \sin(1)} = 0.750364.$$

This should be a better approximation than x_0 to the solution.

For the next approximation

$$x_2 = x_1 - \frac{x_1 - \cos x_1}{1 + \sin x_1} = 0.750364 - \frac{0.750364 - \cos(0.750364)}{1 + \sin(0.750364)} = 0.739113.$$

Then

$$x_3 = x_2 - \frac{x_2 - \cos x_2}{1 + \sin x_2} = 0.739085,$$

and

$$x_4 = x_3 - \frac{x_3 - \cos x_3}{1 + \sin x_3} = 0.739085.$$

Thus the solution is $x = 0.739085$ to six decimal places. Note that we got this on the third step, but we had to go as far as the fourth step to know that it was accurate to six decimal places.

Example Show graphically that the equation $x^3 = \tan^{-1}(x)$ has three solutions, and find an approximation to the positive solution which is correct to four decimal places.

From the graph, it is clear that there are three solutions $x = 0$ and $x = \pm\alpha$. We want to find an approximation to α. In order to be sure that the method finds α and not 0, we'll make sure that our initial guess is bigger than α: let's take $x_0 = 2$.

Write the equation as $f(x) = x^3 - \tan^{-1}(x) = 0$. Then $f'(x) = 3x^2 - \frac{1}{1+x^2}$. So

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - \tan^{-1}(x_n)}{3x_n^2 - \frac{1}{1+x_n^2}}.$$

Thus

$$x_1 = 2 - \frac{2^3 - \tan^{-1}(2)}{3 \cdot 2^2 - \frac{1}{1+2^2}} = 2 - \frac{8 - 1.107149}{12 - \frac{1}{5}} = 2 - \frac{6.892851}{11.8} = 1.415860.$$

Then

$$x_2 = x_1 - \frac{x_1^3 - \tan^{-1}(x_1)}{3x_1^2 - \frac{1}{1+x_1^2}} = 1.084510.$$

$$x_3 = x_2 - \frac{x_2^3 - \tan^{-1}(x_2)}{3x_2^2 - \frac{1}{1+x_2^2}} = 0.937997.$$

$$x_4 = x_3 - \frac{x_3^3 - \tan^{-1}(x_3)}{3x_3^2 - \frac{1}{1+x_3^2}} = 0.903896.$$

$$x_5 = x_4 - \frac{x_4^3 - \tan^{-1}(x_4)}{3x_4^2 - \frac{1}{1+x_4^2}} = 0.902031.$$

$$x_6 = x_5 - \frac{x_5^3 - \tan^{-1}(x_5)}{3x_5^2 - \frac{1}{1+x_5^2}} = 0.902025.$$

Thus the solution is $x = 0.902025$, which is correct to at least 4 decimal places. In fact, $(0.902025)^3 - \tan^{-1}(0.902025) = -0.00000093$.

2.4 Differentiability (8.2.4)

The derivative $f'(a)$ gives the slope of the tangent to the graph $y = f(x)$ at $x = a$. If there is no well-defined tangent at $x = a$, or if $f(x)$ isn't continuous at $x = a$, then we say that $f(x)$ is *not differentiable* at $x = a$. Thus $f(x)$ is *differentiable* at $x = a$ if

a) $f(x)$ is continuous at $x = a$, and

b) The graph of $y = f(x)$ has a well-defined (non-vertical) tangent at $x = a$.

We say that $f(x)$ is *differentiable* if it is differentiable at $x = a$ for every value of a.

Examples $1/x$, $|x|$, $|\sin x|$.

2.5 Higher derivatives (8.3.13)

The derivative $f'(x)$ of a function $f(x)$ is also a function, and may be differentiable itself. Differentiating a function $y = f(x)$ twice yields the *second derivative*, which is written $f''(x)$, $f^{(2)}(x)$, $\frac{d^2f}{dx^2}$, y'' or $\frac{d^2y}{dx^2}$. It tells us the rate of change of $f'(x)$ wrt x: i.e. how the slope of the tangent to $y = f(x)$ is changing as x changes.

Similarly, the second derivative $f''(x)$ may be differentiable, yielding the *third derivative* $f'''(x)$, $f^{(3)}(x)$, or $\frac{d^3f}{dx^3}$. In general, we get the *nth derivative* $f^{(n)}(x)$ or $\frac{d^nf}{dx^n}$ by differentiating $f(x)$ n times in succession.

We say that $f(x)$ is *n times differentiable* if it is possible to differentiate it n times in succession, and that it is *infinitely differentiable* or *smooth* if there is no limit to the number of times it can be differentiated.

Examples

a) Let $f(x) = x^3 + 2x^2 + 3x + 1$. Then $f'(x) = 3x^2 + 4x + 3$, $f''(x) = 6x + 4$, $f'''(x) = 6$, and $f^{(n)}(x) = 0$ for all $n \geq 4$. Thus $f(x)$ is smooth.

b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and so on for ever. Thus $f(x)$ is smooth.

c) Let $f(x) = \frac{1}{x} = x^{-1}$. Then $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$, $f^{(4)}(x) = 24x^{-25}$, and so on. $f(x)$ isn't differentiable at $x = 0$ (since 0 isn't in its maximal domain), but it is smooth everywhere else.

2.6 Maclaurin Series and Taylor Series (9.5.1–2)

Suppose that $f(x)$ is a smooth function, and *suppose* that it can be written as

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \cdots$$

or equivalently as

$$f(x) = \sum_{r=0}^{\infty} a_r x^r.$$

Then we can work out the coefficients a_r by repeatedly differentiating $f(x)$:

$$f(0) = a_0.$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots, \qquad \text{so } f'(0) = a_1.$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots, quad \text{ so } f''(0) = 2a_2 \text{ or } a_2 = f''(0)/2.$$

$$f'''(x) = 6a_3 + 24a_4x + \cdots, \qquad \text{so } f'''(0) = 6a_3 \text{ or } a_3 = f'''(0)/6.$$

$$f^{(4)}(x) = 24a_4 + \cdots, \qquad \text{so } f^{(4)}(0) = 24a_4 \text{ or } a_4 = f^{(4)}(0)/24.$$

In general

$$a_n = f^{(n)}(0)/n!,$$

so

$$f(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

or more concisely

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(x)}{r!}x^r.$$

(where we take 0! to be 1).

This is called the *Maclaurin Series expansion* of $f(x)$. Note that we have simply made the assumption that it is possible to write $f(x)$ in this way: we'll see more later about which functions $f(x)$ this is possible for, and for which values of x it makes sense.

Examples

a) Let $f(x) = x^3 + 2x^2 + 2x + 1$. We have $f(0) = 1$, $f'(x) = 3x^2 + 4x + 2$, so $f'(0) = 2$, $f''(x) = 6x+4$, so $f''(0) = 4$, and $f'''(x) = 6$, so $f'''(0) = 6$. Then $f^{(n)}(x) = 0$ for all $n \geq 4$, so $f^{(n)}(0) = 0$ for all $n \geq 4$. Thus the Maclaurin series expansion is

$$1 + 2x + \frac{4}{2!}x^2 + \frac{6}{3!}x^3 = x^3 + 2x^2 + 2x + 1.$$

Thus for polynomials, we just recover the original polynomial.

b) Let $f(x) = \sin x$. We have $f(0) = 0$, $f'(x) = \cos x$, so $f'(0) = 1$, $f''(x) = -\sin x$, so $f''(0) = 0$, $f'''(x) = -\cos x$, so $f'''(0) = -1$, $f^{(4)}(x) = \sin x$, so $f^{(4)}(0) = 0$, $f^{(5)}(x) = \cos x$, so $f^{(5)}(0) = 1$, and so on for ever. Thus

$$\sin x = \frac{1}{1!}x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{-1}{7!}x^7 + \cdots,$$

or

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

c) Similarly, the Maclaurin series expansion of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Working out the factorials in the series for $\sin x$ we get

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots,$$

and the denominators get small very quickly. If x is also small, then the terms in the Maclaurin series get small very quickly: for example

$$\begin{aligned}\sin(0.1) &= (0.1) - \frac{(0.1)^3}{6} + \frac{(0.1)^5}{120} - \frac{(0.1)^7}{5040} + \cdots \\ &= (0.1) - \frac{1}{6000} + \frac{1}{12000000} - \frac{1}{50400000000} + \cdots\end{aligned}$$

Thus we can get a good approximation to $\sin(0.1)$ by just taking the first few terms.

The first approximation is $\sin(0.1) = 0.1$. The second is $\sin(0.1) = 0.1 - 1/6000 = 0.09983333\ldots$. The third is $\sin(0.1) = 0.1 - 1/6000 + 1/12000000 = 0.09983341666\ldots$, and so on. In fact, $\sin(0.1) = 0.099833416647\ldots$.

Maclaurin's theorem is very good for getting approximations to $f(x)$ when x is very small, but what happens if, for example, we want to get an approximation to $\sin(10)$? The Maclaurin series tells us that

$$\sin(10) = 10 - \frac{10^3}{3!} + \frac{10^5}{5!} - \frac{10^7}{7!} + \frac{10^9}{9!} - \cdots,$$

or

$$\sin(10) = 10 - \frac{1000}{6} + \frac{100000}{120} - \frac{10000000}{5040} + \frac{1000000000}{362880} - \cdots$$

The terms do eventually get small (for example $\frac{10^{35}}{35!} = 0.00012\ldots$), but it takes a long time.

One way to deal with this is to change variable, setting $y = x - a$ for some a, so that when x is close to a, y is close to 0. This change of variable gives the *Taylor series expansion of $f(x)$ about $x = a$:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

or

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is good for approximating $f(x)$ when x is close to a (so that $x - a$ is small).

Examples

a) Let $f(x) = x^3 + x^2 + x + 1$, and let $a = 1$. We have $f(a) = 4$, $f'(x) = 3x^2 + 2x + 1$, so $f'(a) = 6$, $f''(x) = 6x + 2$, so $f''(a) = 8$, and $f'''(x) = 6$, so $f'''(a) = 6$. Then $f^{(n)}(x) = 0$ for all $n \geq 4$, so $f^{(n)}(a) = 0$ for all $n \geq 4$. Thus the Taylor series expansion of $f(x)$ about $x = 1$ is

$$f(x) = 4+6(x-1)+\frac{8}{2!}(x-1)^2+\frac{6}{3!}(x-1)^3 = 4+6(x-1)+4(x-1)^2+(x-1)^3.$$

Thus for a polynomial, we are simply rewriting it as a polynomial in $x - a$.

b) Find an approximation for $f(x) = 1/x$ near $x = 1$ by using the first three terms in the Taylor series expansion.

We have $f(1) = 1$. $f'(x) = -1/x^2$, so $f'(1) = -1$. $f''(x) = 2/x^3$, so $f''(x) = 2$. Hence

$$\frac{1}{x} = 1 - (x-1) + \frac{2}{2!}(x-1)^2 = 1 - (x-1) + (x-1)^2 = x^2 - 3x + 3.$$

Tricks

$x \sin x$, $\sin^2 x$, $\cos^2 x$.

2.7 L'Hopital's rule (9.5.3)

What is $\lim_{x\to 0} \frac{\sin x}{x}$? If we write $\sin x$ as its Maclaurin series expansion, then

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots,$$

and it is obvious that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Similarly, consider $\lim_{x\to 0} \frac{1-\cos x}{x}$. We have

$$\frac{1-\cos x}{x} = \frac{1-(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots}{x} = \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \cdots,$$

and it is obvious that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

We can do the same to work out any limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(a) = g(a) = 0$. Expand $f(x)$ and $g(x)$ as Taylor series about $x = a$ to get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots$$

and so

$$\frac{f(x)}{g(x)} = \frac{f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots} = \frac{f'(a) + \frac{f''(a)}{2!}(x-a) + \cdots}{g'(a) + \frac{g''(a)}{2!}(x-a) + \cdots},$$

from which it is clear that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

This is *L'Hôpital's rule.* **Note** it only works when $f(a) = g(a) = 0$.

If $f'(a) = g'(a) = 0$, then we can extend this to show that the limit is $\frac{f''(a)}{g''(a)}$: if these are both 0, then it is $\frac{f'''(a)}{g'''(a)}$, etc.

Examples

a) What is $\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$? We have $f(x) = x^2 - x - 2$ and $g(x) = x - 2$, so $f'(x) = 2x - 1$ and $g'(x) = 1$. Hence $f'(2) = 3$ and $g'(2) = 1$, so the limit is 3.

b)

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin x - x}{x^3} &= \lim_{x \to 0} \frac{\cos x - 1}{3x^2} \\
&= \lim_{x \to 0} \frac{-\sin x}{6x} \\
&= \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}.
\end{aligned}$$

2.8 The exponential function (2.7.1, 8.3.9)

Functions of the form $f(x) = a^x$, where $a > 1$ is a constant, are called *exponential functions.*

Notice that $a^0 = 1$, $a^1 = a$, a^x is large when x is large, and $a^{-x} = \frac{1}{a^x}$ is small, but positive, when x is large. Thus all of the exponential functions are increasing and have range $(0, \infty)$. Draw graphs of 2^x, 3^x, 4^x.

Exponential functions have the following important properties:

$$\begin{aligned}
a^{x_1} a^{x_2} &= a^{x_1 + x_2} \\
\frac{a^{x_1}}{a^{x_2}} &= a^{x_1 - x_2} \\
a^{kx} &= (a^k)^x = b^x \qquad \text{where } b = a^k.
\end{aligned}$$

By the last property, we only need to understand one exponential function and we understand them all: for example $4^x = 2^{2x}$, $3^x = 2^{kx}$ where k is the number with $2^k = 3$.

We choose a preferred value of a in such a way that a^x *is its own derivative.*

Define the exponential function $f(x) = \exp(x)$ to be the function with $f'(x) = f(x)$ and $f(0) = 1$.

This is enough to tell us its Maclaurin series expansion: since $f^{(n)}(0) = 1$ for all n, we have

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

It can be shown (not hard, but quite a lot of work), that $\exp(x) = e^x$, where

$$e = \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = 2.7182818\ldots$$

.

Thus e^x and $\exp(x)$ are just different ways of writing the same function, which has the crucial property that

$$\frac{d}{dx} e^x = e^x.$$

Draw graph. Note maximal domain is $\mathbf{R}$, range is $(0, \infty)$, increasing, neither even nor odd.

2.9 The logarithmic function (2.7.2, 8.3.9)

The inverse function of $f(x) = a^x$ is called the *logarithm to base a*, written $\log_a$. Thus if $y = a^x$ then $x = \log_a y$.

The inverse of *the* exponential function $y = e^x$ is called the *natural logarithm*, written ln (so ln is just another way of saying $\log_e$). Thus if $y = e^x$ then $x = \ln y$.

We can draw the graph of $y = \ln x$ by using the reflection rule. The maximal domain in $(0, \infty)$, the range is $\mathbf{R}$, and $\ln x$ is increasing.

To differentiate $\ln x$, we use the inverse function rule: if $y = f^{-1}(x)$ (so $x = f(y)$), then $\frac{dy}{dx} = \frac{1}{f'(y)}$. In this case, $y = \ln x$ (so $x = f(y) = e^y$), so

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

The properties of the exponential function give corresponding properties of the logarithm:

$e^{x_1}e^{x_2} = e^{x_1+x_2}$ translates to $\ln(x_1x_2) = \ln x_1 + \ln x_2$.

$\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$ translates to $\ln\frac{x_1}{x_2} = \ln x_1 - \ln x_2$.

$e^{nx} = (e^x)^n$ translates to $\ln x^n = n\ln x$.

Thus for example

$$\begin{aligned}
\ln\left(\frac{\sqrt{10x}}{y^2}\right) &= \ln(\sqrt{10x}) - \ln(y^2) \\
&= \frac{1}{2}\ln(10x) - 2\ln y \\
&= \frac{1}{2}(\ln(10) + \ln x) - 2\ln y.
\end{aligned}$$

$\ln x$ doesn't have a Maclaurin series expansion, since $x = 0$ isn't in the maximal domain of $\ln x$. However, it is possible to calculate the Maclaurin series of $\ln(1+x)$ (this comes down to the same thing as finding the Taylor series of $\ln x$ about $x = 1$).

Have $f(x) = \ln(1+x)$, so $f(0) = \ln(1) = 0$.

$f'(x) = \frac{1}{1+x}$, so $f'(0) = \frac{1}{1} = 1$.

$f''(x) = \frac{-1}{(1+x)^2}$, so $f''(0) = -1$.

$f'''(x) = \frac{2}{(1+x)^3}$, so $f'''(0) = 2$.

$f''''(x) = \frac{-6}{(1+x)^4}$, so $f''''(0) = -6$.

Thus

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

This expansion clearly doesn't make sense if $x \leq -1$, since ln is only defined for $x > 0$. We shall see later that it also doesn't make sense if $x > 1$: in that case, the numerators of the terms grow faster than the denominators. It does, however, work for $x = 1$, when we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

Because the terms get small very slowly, we need to take very many terms to get an accurate approximation to $\ln 2$.

2.10 Hyperbolic functions (2.7.3, 8.3.9)

The *hyperbolic functions* $\sinh x$, $\cosh x$, and $\tanh x$ are defined in terms of e^x: their relationship with the ordinary trig functions will become clear when we do complex numbers.

We have $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, and $\tanh x = \frac{\sinh x}{\cosh x}$. We can also define sech$x$ etc. by analogy with the trig functions.

Notice that $\sinh(-x) = \frac{e - x - e^{-(-x)}}{2} = -\sinh x$, so $\sinh x$ is an odd function. Draw graph. Odd, maximal domain and range are $\mathbf{R}$, increasing.

Similarly $\cosh x$ is even, its maximal domain is $\mathbf{R}$, and its range is $[1, \infty)$.

$\tanh x$ is odd, its maximal domain is $\mathbf{R}$, and its range is $(-1, 1)$.

Notice that $\frac{d}{dx} e^{-x} = -e^{-x}$, so it follows that $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$. By the quotient rule, we have

$$\frac{d}{dx} \tanh x = \frac{d}{dx} \frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = 1 - \tanh^2 x.$$

In fact, there's another way to write $\cosh^2 x - \sinh^2 x$. Notice that $\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x)$. Now $\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$, so $\cosh^2 x - \sinh^2 x = e^x e^{-x} = e^0 = 1$. Compare this with the standard trig identity $\cos^2 x + \sin^2 x = 1$.

In fact, every standard trig identity has a corresponding version for hyperbolic trig functions, which can be obtained by *Osborn's rule*: change the sign of term which involves a product (or implied product) of two sines.

For example $\sin(A + B) = \sin A \cos B + \cos A \sin B$ becomes $\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$. $\cos(A + B) = \cos A \cos B - \sin A \sin B$ becomes $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$. To understand what is meant by an *implied product* the identity $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ becomes $\tanh(A + B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$ since $\tan A \tan B = \frac{\sin A \sin B}{\cos A \cos B}$ involves an implied product of two sines.

Finally, let's work out the Maclaurin series expansions of $\sinh x$ and $\cosh x$. We can either do this directly, by differentiating, or we can note that

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots, \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots. \end{aligned}$$

Thus

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Thus the series for $\cosh x$ consists of the even terms of that for e^x, and the series for $\sinh x$ consists of the odd terms. Compare this with the series expansions of $\sin x$ and $\cos x$.