Chapter 2

Differentiation (8.1–8.3, 9.5)

2.1 Rate of Change (8.2.1–5)

Recall that the equation of a straight line can be written as $y = mx + c$, where m is the *slope* or *gradient* of the line, and c is the *y-intercept* (i.e. the value of y when $x = 0$).

Example $y = 2x + 1$. Draw it. The slope 2 can also be looked on as the *rate of change* of y with respect to x: when x increases by 1, y increases by 2. For example, if x represents time in seconds, and y represents distance travelled in meters, then the rate of change of y with respect to x is the speed of travel.

If the relationship between y and x is more complicated, for example $y = x^2$, then the rate of change of y wrt x is different for different values of x.

Example What is the rate of change of y wrt x when $x = 1$? When $x = 1$, $y = 1$. If x increases by a small amount δ, then y increases to $(1+\delta)^2 = 1+2\delta+\delta^2$, in other words y increases by $2\delta + \delta^2$. Thus

$$\text{Rate of change} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{2\delta + \delta^2}{\delta} = 2 + \delta.$$

To find the instantaneous rate of change at $x = 1$, we let $\delta \to 0$, to obtain 2. Thus the car is travelling at 2 m/s at time 1.

In general, let $y = f(x)$. The rate of change of y with respect to x at $x = x_0$ is given by

$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{\delta \to 0} \frac{f(x_0+\delta) - f(x_0)}{\delta}.$$

Example Return to the example $y = f(x) = x^2$, and let x_0 be any value of x. Then

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{x_0} &= \lim_{\delta \to 0} \frac{f(x_0+\delta) - f(x_0)}{\delta} \\
&= \lim_{\delta \to 0} \frac{(x_0+\delta)^2 - x_0^2}{\delta} \\
&= \lim_{\delta \to 0} \frac{x_0^2 + 2x_0\delta + \delta^2 - x_0^2}{\delta} \\
&= \lim_{\delta \to 0} (2x_0 + \delta) \\
&= 2x_0.
\end{aligned}$$

Thus at time x_0, the speed of the car is $2x_0$. Equivalently, at time x the speed of the car is $2x$. We also write

$$\frac{dy}{dx} = 2x, \qquad y' = 2x, \qquad \frac{df}{dx} = 2x, \qquad \text{or } f'(x) = 2x.$$

The rate of change is called the *derivative of* y *wrt* x, or the *derivative of* $f(x)$ *wrt* x, or just the *derivative of* $f(x)$.

Geometrically $f'(x_0)$ is the slope of the tangent to $y = f(x)$ at $x = x_0$ (picture). Thus the equation of this tangent is $y = f'(x_0)x + c$, where c is the y-intercept. In order to work out c, we use the fact that the tangent passes through the point $(x_0, f(x_0))$. Putting $x = x_0$ and $y = f(x_0)$ in the equation we get $f(x_0) = f'(x_0)x_0 + c$, so $c = f(x_0) - f'(x_0)x_0$, and hence the equation of the tangent is

$$y = f'(x_0)x + f(x_0) - f'(x_0)x_0,$$

or

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Example Find the equation of the tangent to the curve $y = x^2$ at $x_0 = 3$.

When $x_0 = 3$ we have $f(x_0) = 9$, and $f'(x_0) = 2x_0 = 6$. Hence the equation of the tangent is

$$y = 9 + 6(x - 3)$$

or

$$y = 6x - 9.$$

2.2 Derivatives of common functions: rules of differentiation (8.3.1–7)

Recall that if $f(x) = x^2$, then $f'(x) = 2x$. We found this with our bare hands:

$$f'(x) = \lim_{\delta \to 0} \frac{(x+\delta)^2 - x^2}{\delta} = \lim_{\delta \to 0} 2x + \delta = 2x.$$

We can do the same thing for other common functions.

Example Let $f(x) = x^3$. Then

$$\begin{aligned} f'(x) &= \lim_{\delta \to 0} \frac{(x+\delta)^3 - x^3}{\delta} \\ &= \lim_{\delta \to 0} \frac{x^3 + 3x^2\delta + 3x\delta^2 + \delta^3 - x^3}{\delta} \\ &= \lim_{\delta \to 0} (3x^2 + 3x\delta + \delta^2) \\ &= 3x^2. \end{aligned}$$

Thus

$$\frac{d}{dx} x^3 = 3x^2.$$

To find the derivative of x^n for other values of n, we need to be able to work out $(x+\delta)^n$. To do this, we have the *binomial theorem*: to work out $(a+b)^n$, we don't have to work out

$$(a+b)(a+b)(a+b)\ldots(a+b),$$

we can use

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \cdots + \binom{n}{n-1} ab^{n-1} + b^n,$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Rather than work out the coefficients $\begin{pmatrix} n \\ r \end{pmatrix}$ using this formula, we can use *Pascal's triangle*. Draw it. Thus, for example

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Example Expand $(1+2x)^5$ using the binomial theorem.

$$\begin{aligned} (1+2x)^5 &= 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5 \\ &= 1 + 5(2x) + 10(4x^2) + 10(8x^3) + 5(16x^4) + (32x^5) \\ &= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5. \end{aligned}$$

We can use this to work out the derivative of x^n for any n. Let $f(x) = x^n$. Then

$$\begin{aligned} f'(x) &= \lim_{\delta \to 0} \frac{(x+\delta)^n - x^n}{\delta} \\ &= \lim_{\delta \to 0} \frac{x^n + nx^{n-1}\delta + \text{terms in } \delta^2,\, \delta^3 \text{ etc.} - x^n}{\delta} \\ &= \lim_{\delta \to 0} (nx^{n-1} + \text{terms in } \delta,\, \delta^2 \text{ etc.}) \\ &= nx^{n-1}. \end{aligned}$$

Thus

$$\frac{d}{dx} x^n = nx^{n-1}.$$

This gives $\frac{d}{dx}x^2 = 2x$ and $\frac{d}{dx}x^3 = 3x^2$ in agreement with our earlier calculations. We can also now calculate, for example

$$\frac{d}{dx} x^{57} = 57x^{56}.$$

Example Calculate the equation of the tangent to the graph $y = x^{28}$ at $x = 1$.

Write $y = f(x) = x^{28}$. We want to use the formula for the tangent at $x = x_0$:

$$y = f(x_0) + f'(x_0)(x - x_0),$$

so since $x_0 = 1$ the equation is

$$y = f(1) + f'(1)(x - 1).$$

Now $f(1) = 1^{28} = 1$, and $f'(x) = 28x^{27}$, so $f'(1) = 28$. Hence the equation of the tangent is

$$y = 1 + 28(x - 1),$$

or

$$y = 28x - 27.$$

Derivative of $\sin x$ and $\cos x$

Let $f(x) = \sin x$. We can calculate $f'(x)$ using what trigonometric identity (16):

$$\begin{aligned} f'(x) &= \lim_{\delta \to 0} \frac{\sin(x+\delta) - \sin x}{\delta} \\ &= \lim_{\delta \to 0} 2 \frac{\cos\left(\frac{2x+\delta}{2}\right) \sin\left(\frac{\delta}{2}\right)}{\delta} \\ &= \lim_{\delta \to 0} \cos(x + \frac{\delta}{2}) \frac{\sin(\delta/2)}{(\delta/2)} \\ &= \cos x. \end{aligned}$$

Thus $\frac{d}{dx} \sin x = \cos x$.

Similarly $\frac{d}{dx} \cos x = -\sin x$ (exercise).

Example Find the equation of the tangent to the graph $y = \sin x$ at $x = 0$.

Write $f(x) = \sin x$ and $x_0 = 0$. We want to use our formula

$$y = f(x_0) + f'(x_0)(x - x_0)$$

for the equation of the tangent. We have $f(x_0) = \sin 0 = 0$ and $f'(x_0) = \cos 0 = 1$, so the equation is

$$y = 0 + 1(x - 0),$$

or $y = x$.

To find derivatives of other functions, we need some *rules of differentiation*

The constant multiplication rule

If k is a constant, then $\frac{d}{dx} kf(x) = kf'(x)$.

Examples

a) $\frac{d}{dx} 3x^2 = 3(2x) = 6x$.

b) $\frac{d}{dx}5x^4 = 20x^3$.

c) $\frac{d}{dx}2\sin x = 2\cos x$.

The sum rule

If u and v are functions of x, then $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$. Alternatively, $(u+v)' = u' + v'$.

Examples

a) $\frac{d}{dx}(x^3 + 2x + 1) = 3x^2 + 2$. Similarly, we can work out the derivative of any polynomial.

b) $\frac{d}{dx}(x^2 + 2\sin x - \cos x) = 2x + 2\cos x + \sin x$.

The product rule

If u and v are functions of x, then $(uv)' = uv' + u'v$.

Examples

a) Let $f(x) = x^2 \sin x$. We let $u = x^2$ and $v = \sin x$. Thus $u' = 2x$ and $v' = \cos x$. The product rule says that $f'(x) = x^2 \cos x + 2x \sin x$.

b) Let $f(x) = \cos^2 x = \cos x \cos x$. We let $u = v = \cos x$. Then $u' = v' = -\sin x$. The product rule says that $f'(x) = \cos x(-\sin x) + (-\sin x)\cos x = -2\sin x \cos x$. Note $f'(x) = -\sin(2x)$.

c) Let $f(x) = x^2 \sin x \cos x$. We let $u = x^2 \sin x$ and $v = \cos x$. Thus $u' = x^2 \cos x + 2x \sin x$ (part a)), and $v' = -\sin x$. The product rule says that

$$f'(x) = (x^2 \sin x)(-\sin x) + (x^2 \cos x + 2x \sin x)\cos x = x^2(\cos^2 x - \sin^2 x) + 2x \sin x \cos x.$$

(Note $f'(x) = x^2 \cos 2x + x \sin 2x$.)

The quotient rule

If u and v are functions of x, then

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

Examples

a) Let $f(x) = 1/x$. We let $u = 1$ and $v = x$, so $u' = 0$ and $v' = 1$. The quotient rule says that

$$f'(x) = \frac{x(0) - (1)(1)}{x^2} = -1/x^2.$$

b) Let $f(x) = \tan x = \frac{\sin x}{\cos x}$. We let $u = \sin x$ and $v = \cos x$. Thus $u' = \cos x$ and $v' = -\sin x$. Thus

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

c) Let $f(x) = 1/x^n$. We let $u = 1$ and $v = x^n$, so $u' = 0$ and $v' = nx^{n-1}$. The quotient rule says that

$$f'(x) = \frac{x^n(0) - (1)nx^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}}.$$

Written another way,

$$\frac{d}{dx}x^{-n} = -nx^{-n-1},$$

so we can see that

$$\frac{d}{dx}x^n = nx^{n-1}$$

whether n is positive or negative. In fact, we have $\frac{d}{dx}x^a = ax^{a-1}$ for *any* number a. Some examples:

d) Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = (1/2)x^{-1/2} = \frac{1}{2\sqrt{x}}$.

e) Let $f(x) = \frac{1}{\sqrt[3]{x}} = x^{-1/3}$. Then $f'(x) = -(1/3)x^{-4/3} = \frac{-1}{3x\sqrt[3]{x}}$.

The chain rule

Let $f(x) = g(h(x))$. Then $f'(x) = g'(h(x))h'(x)$.

Examples

a) Let $f(x) = (4x-1)^3$. Let $g(x) = x^3$ and $h(x) = 4x - 1$, so $f(x) = g(h(x))$. We have $g'(x) = 3x^2$ and $h'(x) = 4$. Thus

$$f'(x) = g'(h(x))h'(x) = 3(4x-1)^2 \cdot 4 = 12(4x-1)^2.$$

b) Let $f(x) = \sin(3x+2)$. Let $g(x) = \sin x$ and $h(x) = 3x+2$, so $f(x) = g(h(x))$. We have $g'(x) = \cos x$ and $h'(x) = 3$. Thus

$$f'(x) = g'(h(x))h'(x) = \cos(3x+2) \cdot 3 = 3\cos(3x+2).$$

More generally, $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$ and $\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$.

c) Let $f(x) = (\sin x + \cos 3x)^3$. Let $g(x) = x^3$ and $h(x) = \sin x + \cos 3x$, so $f(x) = g(h(x))$. We have $g'(x) = 3x^2$ and $h'(x) = \cos x - 3\sin 3x$. Thus

$$f'(x) = g'(h(x))h'(x) = 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x).$$

d) Let $f(x) = \tan((\sin x + \cos 3x)^3)$. Let $g(x) = \tan x$ and $h(x) = (\sin x + \cos 3x^3)$, so $f(x) = g(h(x))$. We have $g'(x) = \sec^2(x)$ and $h'(x) = 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x)$, so

$$f'(x) = g'(h(x))h'(x) = \sec^2((\sin x + \cos 3x)^3) \cdot 3(\sin x + \cos 3x)^2(\cos x - 3\sin 3x).$$

The Inverse Function Rule

Let $y = f^{-1}(x)$ (so $x = f(y)$). Then

$$\frac{dy}{dx} = \frac{1}{f'(y)}.$$

Examples

a) Let $y = \sqrt{x}$ (so $x = y^2$, and we have $f(y) = y^2$. Then

$$\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}.$$

Thus

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

This agrees with our earlier way of calculating this: $\frac{d}{dx}x^{1/2} = frac12x^{-1/2}$.

b) Let $y = \sin^{-1}(x)$ (so $x = \sin y$, and we have $f(y) = \sin y$. Then

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

Thus

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}.$$

c) Similarly, it can be shown that

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1 - x^2}}.$$

d) Let $y = \tan^{-1}(x)$ (so $x = \tan y$, and we have $f(y) = \tan y$. Then

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

Thus

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1 + x^2}.$$