MATH105 Numbers and Sets

Basic propositional logic is simply a shorthand used in writing mathematics. The basic symbols are

$\vee$ or

$\wedge$ and

$\Longrightarrow$ "If .. then", that is $A \Rightarrow B$ means "If A then B"

$\Longrightarrow$ can be interpreted as "implies" or "only if"

$\Longleftarrow$ "if" or "is implied by" $A \Leftarrow B$ means "A if B"

Examples If x is a real number, $x > 0 \vee x < 0 \vee x = 0$ True

" " $x > 0$ or $x < 0$ or $x = 0$

" " $x \geqslant 0 \vee x \leq 0$ True

" " $x \geqslant 0$ ~~or~~ $x \leq 0$

$(1 < 2) \wedge (2 < 3)$ $1 < 2$ and $2 < 3$ True

If x, y, z are real numbers, $(x<y) \wedge (y<z) \Rightarrow x < z$ True

" " " and $x<y$ and $y<z$, then $x<z$

In all the following statements, x is a real number

$x^2 = 1 \Longleftrightarrow (x = 1 \vee x = -1)$ True

$x^2 = 1$ if and only if $x = 1$ or $x = -1$

$x = 1 \Longrightarrow x^2 = 1$ True

If $x = 1$ then $x^2 = 1$

Which of the following statements are true?

Once again, x is a real number.

$x^2 = 4 \Rightarrow x = 2$ False

$(2 < 3) \vee (3 < 2)$ True

$x^2 = x \Leftrightarrow x^2 - x = 0$ True

$x = 1 \Rightarrow x^2 = x$ True

$x^2 = x \Rightarrow x = 1$ False

$x^2 = x \Rightarrow (x = 0 \vee x = 1)$ ~~False~~ True

$(x = 0 \vee x = 1) \Rightarrow x^2 = x$ True

$x > 1 \Rightarrow x > 0$ True

$x > 1 \Leftrightarrow -x > -1$ False.

Negation

If A is a statement then $\neg A$ is the statement "not A"

Examples If A is "It will rain today" then $\neg A$ is "It will not rain today"

In the following examples, x, y and z are real numbers

$\neg(x < y)$ is $x \geq y$ - which can also be written as $y \leq x$

$\neg(x \leq y)$ is $y < x$ - which can also be written as $x > y$

What about $\neg(x < y < z)$?

$x < y < z$ is the same as $\overset{A}{(x < y)} \wedge \overset{B}{(y < z)}$

For this to be not true, either $x \geq y$ or $y \geq z$ or both

So $\neg(x<y<z)$ is $(\underbrace{x \geq y}_{\neg A}) \vee (\underbrace{y \geq z}_{\neg B})$

In <u>general</u> $\neg(A \wedge B)$ is $\neg A \vee \neg B$

What about $\neg(x>3 \vee x<-1)$?

It is not true that ~~$x<3$ or x~~ $x>3$ or $x<-1$ if and only if

$-1 \leq x \leq 3$

So $\neg(\underbrace{x>3}_{A} \vee \underbrace{x<-1}_{B})$ is $-1 \leq x \leq 3$, which can also be written as $\underbrace{x \leq 3}_{\neg A} \wedge \underbrace{-1 \leq x}_{\neg B}$

<u>In general</u> $\neg(A \wedge B)$ is $\neg A \vee \neg B$

<u>Theorems</u>

A <u>theorem</u> is a <u>statement</u> which is <u>true</u>.

<u>Example</u> $2<3$ is a theorem. ~~2<1 is~~ $2<1$ is <u>not</u> a theorem

$x^2>4 \Leftrightarrow (x>2 \vee x<-2)$ is a theorem.

If we want to prove a theorem C, we might start with a theorem A which we know to be true and try to deduce C from it.

If A is true and $A \Rightarrow C$ then C is true

If A is true and $A \Rightarrow B$ and $B \Rightarrow C$ ~~is true~~ then C is true

We could have a longer chain of implications.
e.g. if $A \Rightarrow B_1$, $B_1 \Rightarrow B_2$, $B_2 \Rightarrow C$ and A is true, then C is true.

It is a good idea to use $\Leftrightarrow$ wherever possible

Examples

Theorem $x^2 > 4 \Leftrightarrow x > 2 \vee x < -2$

Proof $x^2 > 4 \Leftrightarrow x^2 - 4 > 0 \Leftrightarrow (x-2)(x+2) > 0$

$\Leftrightarrow (x-2 > 0 \wedge x+2 > 0) \vee (x-2 < 0 \wedge x+2 < 0)$

(This is because a product of 2 real numbers is > 0 if and only if both numbers are > 0 or both < 0)

$\Leftrightarrow (x > 2 \wedge x > -2) \vee (x < 2 \wedge x < -2)$

$\Leftrightarrow x > 2 \vee x < -2.$ $\square$

Theorem $x^2 - 3x + 2 = 0 \Rightarrow x = 1 \vee x = 2$

Proof $x^2 - 3x + 2 = 0 \Leftrightarrow (x-1)(x-2) = 0 \Leftrightarrow$
$x - 1 = 0 \vee x - 2 = 0 \Leftrightarrow x = 1 \vee x = 2$ $\square$

We used $\Leftrightarrow$ throughout which is stronger than $\Rightarrow$.

Theorem If x is a real number
$x^2 - 3x + 2 \leq 0 \Leftrightarrow 1 \leq x \leq 2$

Proof $x^2 - 3x + 2 \leq 0 \Leftrightarrow (x-1)(x-2) \leq 0$

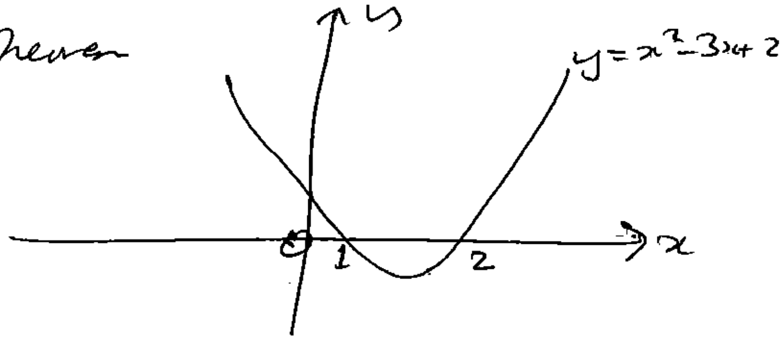
$\Leftrightarrow ((x-1 \geq 0 \wedge x-2 \leq 0) \vee (x-1 \leq 0 \wedge x-2 \geq 0))$

(the product of 2 real numbers is $\leq 0 \Leftrightarrow$ one of them is ≥ 0 and one ≤ 0)

$\Leftrightarrow ((x \geq 1 \wedge x \leq 2) \vee (x \leq 1 \wedge x \geq 2))$

$\Leftrightarrow 1 \leq x \leq 2$ because no real number x satisfies $x \leq 1 \wedge x \geq 2$ $\square$

A graph confirms this theorem



Theorem If x is a real number,

$x^2-3x+2 > 12 \iff (x<-2 \vee x>5)$

Proof $x^2-3x+2>12 \iff x^2-3x-10>0$

$\iff (x+2)(x-5)>0$

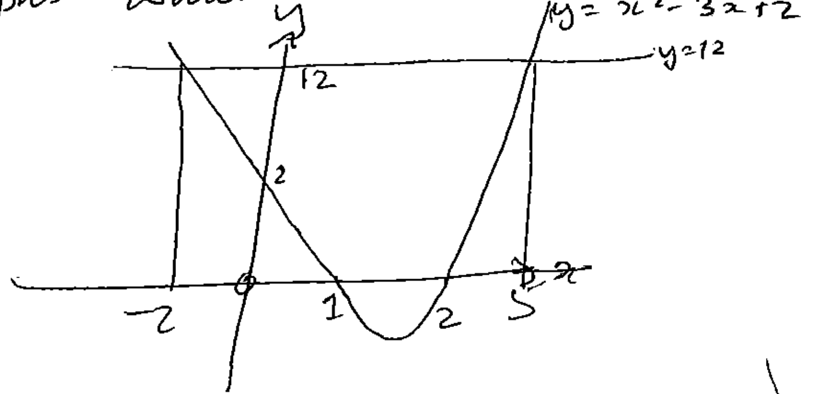
$\iff (x+2>0 \wedge x-5>0) \vee (x+2<0 \wedge x-5<0)$

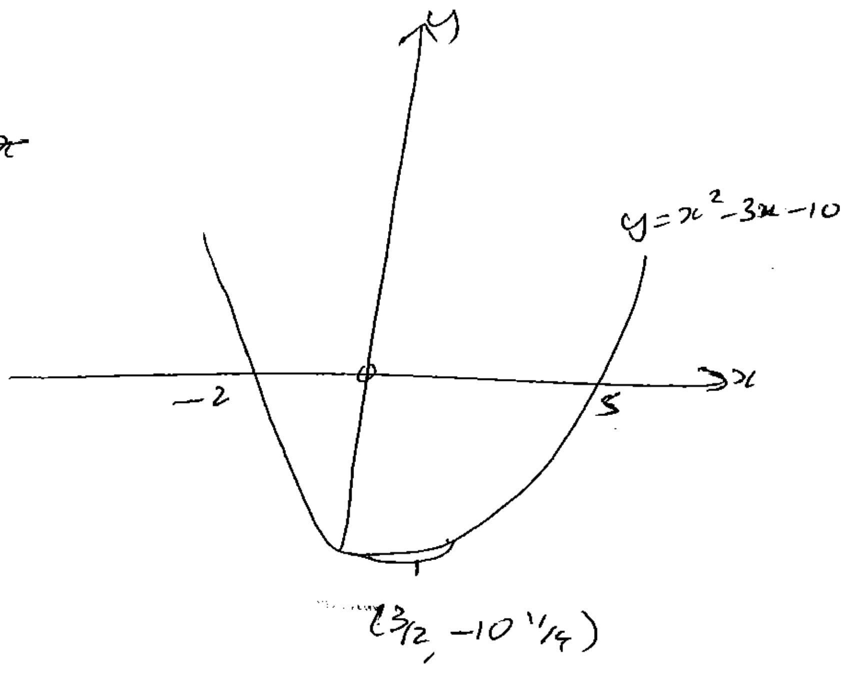
$\iff (x>-2 \wedge x>5) \wedge (x<-2 \wedge x<5)$

$\iff x>5 \vee x<-2$

$\iff x<-2 \vee x>5$ $\square$

A graph confirms this theorem. Actually, there are two graphs which one can naturally draw





Theorem If x is a real number, $x \neq -1$, $\left|\frac{1}{x+1}\right| < 1 \Leftrightarrow (x > 0 \vee x < -2)$

Proof $\left|\frac{1}{x+1}\right| < 1 \Leftrightarrow \frac{1}{(x+1)^2} < 1$ (Modulus is always ≥ 0. The square of a number ≥ 0 is $< 1 \Leftrightarrow$ the number itself is < 1. Also $y^2 = (-y)^2$ for all real numbers y.)

$\Leftrightarrow 1 < (x+1)^2$ (Inequalities are preserved when multiplying through by numbers > 0 and $(x+1)^2 > 0$ because we know $(x+1)^2 \neq 0$ [illegible])

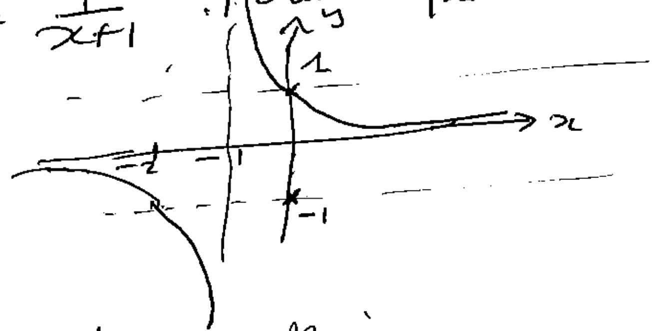
$\Leftrightarrow 1 < x^2 + 2x + 1 \Leftrightarrow 0 < x^2 + 2x \Leftrightarrow 0 < x(x+2)$

$\Leftrightarrow (0 < x \wedge 0 < x+2) \vee (x < 0 \wedge x+2 < 0)$

$\Leftrightarrow (0 < x \wedge -2 < x) \vee (x < 0 \wedge x < -2)$

$\Leftrightarrow 0 < x \vee x < -2$ $\square$

Again, this is confirmed by drawing a graph. The easiest graph to draw is probably $y = \frac{1}{x+1}$. Note that $\left|\frac{1}{x+1}\right| < 1 \Leftrightarrow -1 < \frac{1}{x+1} < 1$.



Theorem If x and y are real numbers then
$x^2 + xy + y^2 \leq 0 \Leftrightarrow x = y = 0$

Proof $x^2 + xy + y^2 \leq 0 \Leftrightarrow (x + \frac{1}{2}y)^2 + \frac{3}{4}y^2 \leq 0$ (A square of any real number is ≥ 0 and $= 0 \Leftrightarrow$ the number $= 0$)

$\Leftrightarrow (x+\frac{1}{2}y)^2 + \frac{3}{4}y^2 = 0$

$\Leftrightarrow x + \frac{1}{2}y = 0 \wedge y = 0$

$\Leftrightarrow x = 0 \wedge y = 0$ $\square$

Equivalent ~~Negating~~ implications

If A and B are statements, then $A \Rightarrow B$ is equivalent to $\neg B \Rightarrow \neg A$.

Example Suppose x is a real number. Let A be the statement $x < 2$ and let B be the statement $x < 3$

$A \Rightarrow B$, that is $x < 2 \Rightarrow x < 3$ is a true statement

$\neg A$ is $x \geqslant 2$ $\quad \neg B$ is $x \geqslant 3$

$x \geqslant 3 \Rightarrow x \geqslant 2$ $\quad$ So we see in this example that $\neg B \Rightarrow \neg A$

$\neg B \Rightarrow \neg A$

But $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ holds whatever the statements A and B are.

Example Again, suppose x is a real number.
Let A be the statement $x^2 < 4$ $\quad$ Let B be $x < 2$

$x^2 < 4 \Rightarrow x < 2$ $\quad$ is a true statement

$A \Rightarrow B$

$\neg A$ is $x^2 \geqslant 4$ $\quad \neg B$ is $x \geqslant 2$

$\neg B \Rightarrow \neg A$ $\quad x \geqslant 2 \Rightarrow x^2 \geqslant 4$ is true.

Note that $\neg A$ does not imply $\neg B$

$x^2 \geqslant 4$ does not imply $x \geqslant 2$

e.g. $x = -3$ satisfies $(-3)^2 \geqslant 4$ and ~~but~~ $\neg(-3 \geqslant 2)$

$-3 < 2$

Example Let x and y be real numbers.
Let A be $x > 0 \wedge y > 0$ $\quad$ Let B be $xy > 0$

$A \Rightarrow B$. $\quad (x > 0 \wedge y > 0) \Rightarrow xy > 0$

$\neg B$ is $xy \leqslant 0$ $\quad \neg A$ is $(x \leqslant 0) \vee (y \leqslant 0)$

$\neg B \Rightarrow \neg A$ $\quad xy \leqslant 0 \Rightarrow (x \leqslant 0 \vee y \leqslant 0)$

But it is not true that $\neg A \Rightarrow \neg B$.