Rational Numbers

All numbers are constructed out of the integers, which are themselves constructed from the natural numbers.

The rational numbers are constructed using an equivalence relation on pairs of integers.

$$X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{(m_1, m_2) : m_1, m_2 \in \mathbb{Z},\ m_2 \neq 0\}$$

Define $(m_1, m_2) \sim (n_1, n_2) \iff m_1 n_2 = m_2 n_1$

$(m_1, m_2) \sim (m_1, m_2)$ $\forall (m_1, m_2) \in X$, because $m_1 m_2 = m_2 m_1$

$\sim$ is reflexive

$\forall (m_1, m_2)$ and $(n_1, n_2) \in X$,
$(m_1, m_2) \sim (n_1, n_2) \iff m_1 n_2 = m_2 n_1 \iff n_1 m_2 = n_2 m_1$
$\iff (n_1, n_2) \sim (m_1, m_2)$ So $\sim$ is symmetric

$\forall (m_1, m_2), (n_1, n_2), (p_1, p_2) \in X$,

$\big((m_1, m_2) \sim (n_1, n_2) \wedge (n_1, n_2) \sim (p_1, p_2)\big) \iff (m_1 n_2 = m_2 n_1 \wedge n_1 p_2 = n_2 p_1)$

$\implies (m_1 n_2 p_2 = m_2 p_2 n_1 \wedge m_2 n_1 p_2 = m_2 n_2 p_1)$

$\implies (m_1 n_2 p_2 = m_2 n_2 p_1) \implies (m_1 p_2 = m_2 p_1$ because $n_2 \neq 0)$

$\implies (m_1, m_2) \sim (p_1, p_2)$

So $\sim$ is transitive

So $\sim$ is an equivalence relation.

We denote the equivalence class of (m_1, m_2) by $\frac{m_1}{m_2}$

By the definition of $\sim$ $\quad \frac{m_1}{m_2} = \frac{n_1}{n_2} \iff m_1 n_2 = n_1 m_2$

Definition A rational number is an equivalence class $\frac{m_1}{m_2}$ for some $(m_1, m_2) \in X$. The set of rational numbers is denoted by $\mathbb{Q}$

e.g. $\frac{0}{1} = 0 \quad \frac{3}{7} = \frac{6}{?} = \frac{-3}{?}$

Addition in $\mathbb{Q}$

If $\frac{m_1}{m_2}$ and $\frac{n_1}{n_2} \in \mathbb{Q}$ then we define

$$\frac{m_1}{m_2} + \frac{n_1}{n_2} = \frac{m_1 n_2 + n_1 m_2}{m_2 n_2}$$

This is well defined. For suppose $\frac{m_1}{m_2} = \frac{m_1'}{m_2'}$ and $\frac{n_1}{n_2} = \frac{n_1'}{n_2'}$

Then $m_1 m_2' = m_1' m_2$ and $n_1 n_2' = n_1' n_2$

Then $$\frac{m_1 n_2 + n_1 m_2}{m_2 n_2} = \frac{m_1 n_2 m_2' n_2' + n_1 m_2 m_2' n_2'}{m_2 n_2 m_2' n_2'}$$

$$= \frac{m_1' m_2 n_2 n_2' + n_1' m_2 m_2' n_2}{m_2 n_2 m_2' n_2'} = \frac{m_1' n_2' + n_1' m_2'}{m_2' n_2'}$$

Also if $q = \mathrm{lcm}(m_2, n_2)$ and $q = m_2 p$ and $q = n_2 r$ for $p, r \in \mathbb{Z} \setminus \{0\}$ we have

$$\frac{m_1}{m_2} + \frac{n_1}{n_2} = \frac{m_1 n_2 + n_1 m_2}{m_2 n_2} = \frac{m_1 n_2 p r + n_1 m_2 p r}{(m_2 p)(n_2 r)}$$

$$= \frac{m_1 p q + n_1 r q}{q^2} = \frac{m_1 p + n_1 r}{q} = \frac{m_1 p}{q} + \frac{n_1 r}{q}$$

Example $$\frac{2}{15} + \frac{7}{10} = \frac{4}{30} + \frac{21}{30} = \frac{25}{30} = \frac{5}{6}$$

With this definition of addition, $\frac{m}{1} + \frac{n}{1} = \frac{m+n}{1}$, which is what we want, as we want to identify $\frac{m}{1}$ with $m \in \mathbb{Z}$.

Also, for any $q \in \mathbb{Z} \setminus \{0\}$, we have $\frac{m}{q} + \frac{n}{q} = \frac{m+n}{q}$

Multiplication in $\mathbb{Q}$

For $\frac{m_1}{m_2}, \frac{n_1}{n_2} \in \mathbb{Q}$ we define

$$\frac{m_1}{m_2} \cdot \frac{n_1}{n_2} = \frac{m_1 n_1}{m_2 n_2}$$

This is, once again, well-defined.

All the usual arithmetic laws hold for $\mathbb{Q}$ - associativity, commutativity, distributivity - just as described in the notes on natural numbers

Subtraction

Defn $\frac{m_1}{m_2} - \frac{n_1}{n_2} := \frac{m_1 n_2 - n_1 m_2}{m_2 n_2}$

is defined to be

Division

By definition $\frac{m_1}{m_2} = 0 \iff m_1 = 0$

If $\frac{n_1}{n_2} \neq 0$ then $\frac{m_1}{m_2} \div \frac{n_1}{n_2} := \frac{m_1 n_2}{m_2 n_1} = \frac{m_1}{m_2} \cdot \frac{n_2}{n_1}$

So if $\frac{m_1}{m_2} \neq 0$, $\frac{m_1}{m_2} \div \frac{m_1}{m_2} = \frac{m_1 m_2}{m_1 m_2} = \frac{1}{1} = 1$

~~Example~~

Example

Find $a, b, c \in \mathbb{Z}$ such that

$$a \cdot \frac{2}{9} + b \cdot \frac{1}{6} + c \cdot \frac{10}{21} = \frac{1}{N} \quad \text{where} \quad N = \operatorname{lcm}(9, 6, 21)$$

As $9 = 3^2 \quad 6 = 2 \times 3 \quad 21 = 3 \times 7$

$N = 3^2 \times 2 \times 7 = 126$

$\operatorname{lcm}(9, 6) = 18.$

To find a, b, c we can find $a_1, b_1, a_2, b_2 \in \mathbb{Z}$ so that

$$a_1 \cdot \frac{2}{9} + b_1 \cdot \frac{1}{6} = \frac{1}{18} \quad \text{and} \quad a_2 \cdot \frac{1}{18} + b_2 \cdot \frac{10}{21} = \frac{1}{126}$$

$$\frac{4a_1 + 3b_1}{18} = \frac{1}{18} \qquad 4a_1 + 3b_1 = 1 \quad a_1 = -2 \quad b_1 = 3$$

Then $7a_2 + 60 b_2 = 1$

$$\left(\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 60 \end{array}\right) \xrightarrow{R_2 - 8R_1} \left(\begin{array}{cc|c} 1 & 0 & 7 \\ -8 & 1 & 4 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 9 & -1 & 3 \\ -8 & 1 & 4 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 9 & -1 & 3 \\ -17 & 2 & 1 \end{array}\right)$$

$a_2 = -17 \quad b_2 = 2$

$$-17\left(-2 \cdot \frac{2}{9} + 3 \cdot \frac{1}{6}\right) + 2 \cdot \frac{10}{21} = \frac{1}{126}$$

$$34 \cdot \frac{2}{9} - 51 \cdot \frac{1}{6} + 2 \cdot \frac{10}{21} = \frac{1}{126}$$

$a = 34 \quad b = -51 \quad c = 2$

In general if $\operatorname{lcm}(m, n) = q$ then for any $p \in \mathbb{Z}$ $\frac{p}{q}$ can be written in the form $\frac{a}{m} + \frac{b}{n}$ for $a, b \in \mathbb{Z}$. This follows from the form taken by the gcd for any 2 non zero integers.

Order in $\mathbb{Q}$

Any element of $\mathbb{Q}$ can be written as $\frac{m_1}{m_2}$ with $m_1 \in \mathbb{Z}$ and $m_2 \in \mathbb{Z}_+$. If $\frac{m_1}{m_2}$ and $\frac{n_1}{n_2}$ have $m_1, n_1 \in \mathbb{Z}$ and $m_2, n_2 \in \mathbb{Z}_+$ then we define

$$\frac{m_1}{m_2} < \frac{n_1}{n_2} \iff m_1 n_2 < m_2 n_1$$

Thus order in $\mathbb{Q}$ is defined in terms of order in $\mathbb{Z}$.

All the usual properties of order ~~that~~ that hold for $\mathbb{N}$ or $\mathbb{Z}$ also hold for $\mathbb{Q}$

e.g. $x < y \iff x+z < y+z \quad \forall x, y, z \in \mathbb{Q}$

So $x < y \iff 0 < y - x$

If $z > 0$ then $x < y \Rightarrow zx < zy \quad \forall x, y, z \in \mathbb{Q}$

<u>One different property</u> $\forall x, y \in \mathbb{Q}$ with $x < y$, $\exists z \in \mathbb{Q}$ with $x < z < y$.

One way to do this is to take $z = \frac{x+y}{2}$

Then $z - x = y - z = \frac{y-x}{2} > 0$ if $y > x$

So $x < y \Rightarrow x < z < y$