Algebraic numbers.

Defn $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) is <u>algebraic</u> if there is a polynomial $f(x) = a_0 + a_1 x \cdots + a_n x^n$, with $a_i \in \mathbb{Z}$ for $0 \le i \le n$, such that $f(\alpha) = 0$

Examples $\sqrt{2}$ is algebraic, because $(\sqrt{2})^2 - 2 = 0$

$-\frac{1 \pm \sqrt{5}}{2}$ ~~are~~ algebraic because these are the zeros of $x^2 + x - 1$.

$2^{1/3}$ is algebraic (where this could mean the real root, or either ~~(of this~~ of the complex roots) because $(2^{1/3})^3 - 2 = 0$; ~~any~~ $2^{1/3}$ is a zero of the polynomial $x^3 - 2$.

The zeros of $x^3 + x + 1$ are algebraic...

Theorem The set of algebraic numbers is countable.

This uses another theorem

Theorem If A_i is countable for all $i \in \mathbb{Z}_+$ then $\bigcup_{i=1}^{\infty} A_i$ is countable

Proof ~~Let $f_i : \mathbb{Z}_+ \longrightarrow A_i$ be~~ a bijection ~~countable~~

~~then $f : \mathbb{Z}_+ \times \mathbb{Z}_+$ defined by~~

We can assume the A_i are all disjoint because

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(A_i \setminus \bigcup_{j=1}^{i-1} A_j\right)$$

We can also assume $A_i \neq \emptyset$ for all i

Then let $f_i : A_i \longrightarrow \mathbb{Z}_+$ be injective

(A_i could be finite)

Then $f : \bigcup_{i=1}^{\infty} A_i \longrightarrow \mathbb{Z}_+ \times \mathbb{Z}_+$ defined by

$$f(a) = (i, f_i(a)) \quad \text{if } a \in A_i$$

is injective.

The set of polynomials $a_0 + a_1 x + \cdots + a_n x^n$ with $a_i \in \mathbb{Z}$, $a_n \neq 0$ is countable.

So the set of <u>all</u> polynomials with integer coefficients is a countable union of countable sets, hence countable.

The set of all algebraic numbers is the union of the finite sets of zeros of polynomials with integer coefficients hence countable.

<u>Defn</u> A real (or complex) number is <u>transcendental</u> if it is not algebraic.

<u>Examples</u> π and e are transcendental.

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Complex numbers.

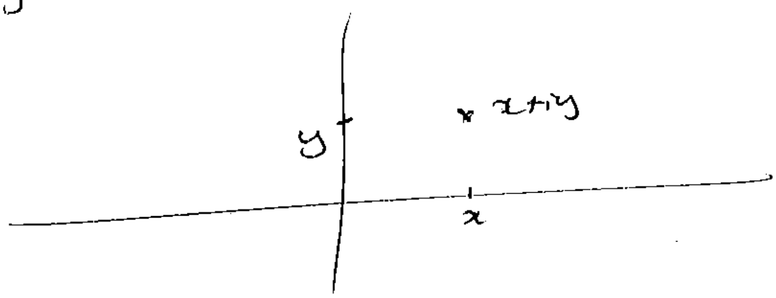
A complex number is ~~a no~~ written in the form $x+iy$ for $x, y \in \mathbb{R}$, where we write $x + i0 = x$, so $\mathbb{R} \subset \mathbb{C}$

$$\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$$

Addition, subtraction and multiplication, $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$ are defined for any $z_1, z_2 \in \mathbb{C}$.

Addition and multiplication are commutative. There is also a distributive law, so that from $i.i = i^2 = -1$ we can define $(x_1+iy_1)(x_2+iy_2)$ for any $x_1, y_1, x_2, y_2 \in \mathbb{R}$, just by "multiplying out the brackets"

Complex numbers identify with points in the plane

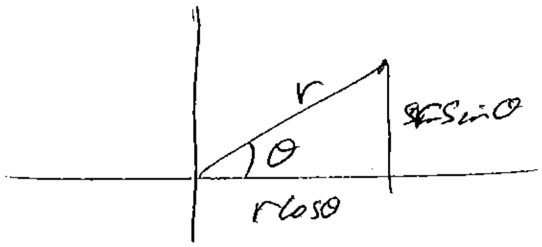


Apart from that, what ~~distin~~ is the advantage of complex numbers over real numbers?

We know that -1 has complex roots, because $(i)^2 = (-i)^2 = -1$ and real square roots of -1 do not exist.

More generally, any non zero complex number has n distinct n'th roots.

This ~~arises~~ is seen by looking at complex multiplication in polar form.



$$(r\cos\theta + i\sin\theta)^n = r^n\cos n\theta + r^n\sin n\theta$$

$$(r\,\mathrm{cis}\,\theta)^n = r^n\,\mathrm{cis}\,n\theta \qquad \text{where } \mathrm{cis}\,\theta = \cos\theta + i\sin\theta.$$

We also write $\cos\theta + i\sin\theta = e^{i\theta}$ so that

$$(re^{i\theta})^n = r^n e^{in\theta}$$

This is more than a notational device, but we could regard it as just that. More than this is true

<u>Fundamental Theorem of algebra</u>

Let $f(z) = a_n z^n + \dots + a_0$ with $a_i \in \mathbb{C}$, $0 \le i \le n$, and $n > 0$, $a_n \neq 0$. Then $f(z) = a_n \prod_{i=1}^{n} (z - \alpha_i)$ for $\alpha_i \in \mathbb{C}$, $1 \le i \le n$

Hence $f(\alpha) = 0 \iff \alpha = \alpha_i$ for some $1 \le i \le n$

So f has n zeros "up to multiplicity".

<u>Notes on proof</u>

Note that for any polynomial f with complex coefficients and any $\alpha \in \mathbb{C}$, $f(\alpha) = 0 \iff f(z) = (z - \alpha)g(z)$ for some polynomial α.

For suppose $f(z) = a_n z^n + \dots + a_0$. We can assume $a_n \neq 0$

$f(z) = (z - \alpha)g(z) + c$ for some polynomial g and $c \in \mathbb{C}$, and $f(\alpha)=0 \Leftrightarrow c=0$.

To write f in this form we need to solve

$$a_n z^n + \dots + a_0 = (z-\alpha)(b_{n-1}z^{n-1} + \dots + b_0) + c$$

So we need $b_{n-1} = a_n$ if $n>0$

$g=0$ and $c = a_n$ if $n=0$

for $0 \le m < n-1$ $\quad -\alpha b_{m+1} + b_m = a_{m+1}$ gives coeff. of z^{m+1}

$$a_0 = -\alpha b_0 + c$$

The Fundamental Theorem of Algebra is proved by induction on n. Base case $\quad a_1 z + a_0 = a_1\left(z + \frac{a_0}{a_1}\right)$ if $a_1 \neq 0$

Inductive step Suppose $n>1$ and the theorem is proved for $n-1$. It suffices to show $\exists\, \alpha \in \mathbb{C}$ with $f(\alpha)=0$ for then $f(z) = (z-\alpha)g(z)$ where g has degree $n-1$ and we can use the inductive hypothesis on g.

This is a theorem which is not true for real numbers. What makes it work for complex numbers is essentially the existence of complex roots.

Proof of the inductive step of the Fundamental Theorem of Algebra

Let $f(z) = a_n z^n + \dots + a_0$

We want to prove $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$.

Suppose not. Then we will obtain a contradiction.

$f(z) \neq 0 \quad \forall z \in \mathbb{C} \Rightarrow \dfrac{1}{f(z)}$ is continuous on $\mathbb{C}$

Put $R \geq 2\left(1 + \dfrac{|a_0| + \dots + |a_{n-1}|}{|a_n|}\right)$

If $|z| = R$ $\quad |f(z)| \geq |a_n||z|^n - |z|^{n-1}(|a_0| + \dots + |a_{n-1}|)$

$= |z|^{n-1}\left(|a_n||z| - (|a_0| + \dots + |a_{n-1}|)\right) \geq \frac{1}{2}|a_n||z|^n$

So if $|z| = R$ $\quad \left|\dfrac{1}{f(z)}\right| \leq \dfrac{2}{|a_n||z|^n}$

So if R is large enough, the maximum of $\dfrac{1}{|f(z)|}$ on $\{z : |z| \leq R\}$ does not occur on $\{z : |z| = R\}$, and must occur at some point z_0 with $|z_0| < R$. So there is $z_0 \in \mathbb{C}$ such that

$|f(z)| \geq |f(z_0)| \quad \forall\, |z - z_0| \leq \delta$, some $\delta > 0$.

Write $f(z) = b_n(z - z_0)^n + \dots + b_0$ some $b_i \in \mathbb{C}$ $\quad b_0 \neq 0$

So $f(z_0) = b_0$

Let m be the least integer ≥ 1 with $b_m \neq 0$

$$f(z) = b_0 + b_m(z - z_0)^m\left(1 + \frac{b_{m+1}}{b_m}(z - z_0) \dots + \frac{b_n}{b_m}(z - z_0)^{n-m}\right)$$

Let $\delta \leq \frac{1}{2}\left(\dfrac{|b_m|}{1 + |b_{m+1}| \dots + |b_n|}\right)$

Then if $|z - z_0| \leq \delta$, $\quad |f(z) - b_0 - b_m(z - z_0)^m| \leq \frac{1}{2}|b_m||z - z_0|^m$

Then $|f(z)| \leq |b_0| - \frac{1}{2}|b_m|\delta^m$ if $z - z_0 = \delta e^{i\alpha}$ $\quad b_0 = |b_0|e^{i\beta}$

$b_m = |b_m|e^{i\gamma}$ and $m\alpha + \gamma = \pi + \beta$ $\quad$ X