Cardinality

Back to sets!

<u>Definition</u> We say that sets A and B <u>have the same cardinality</u> if there is a bijection $f: A \to B$

If $f: A \to B$ is a bijection then so is $f^{-1}: B \to A$

If $f: A \to B$ and $g: B \to C$ are bijections then so is $g \circ f: A \to C$. So "having the same cardinality" is an equivalence relation on any set of sets

<u>Examples</u> $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$ all have the same cardinality

$g: \mathbb{N} \to \mathbb{Z}_+$ defined by $g(n) = n+1$ is a bijection.

$f: \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -\frac{(n+1)}{2} & \text{if } n \text{ odd} \end{cases}$

is a bijection with $f^{-1}(m) = \begin{cases} 2m & \text{if } m \in \mathbb{N} \\ -2m-1 & \text{if } m \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$

This is straightforward. But more surprisingly:

<u>Theorem 1</u> $\mathbb{Z}$ and $\mathbb{Q}$ (and $\mathbb{Z}_+$ and $\mathbb{N}$) have the same cardinality

<u>Theorem 2</u> $\mathbb{R}$ does <u>not</u> have the same cardinality as $\mathbb{Q}$ (and $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$)

In order to prove these theorems, another theorem is used.

<u>Schroeder Bernstein Theorem</u> Let A, B be sets and suppose there are injective maps $f: A \longrightarrow B$ and $g: B \to A$. Then there is a bijection $F: A \longrightarrow B$

<u>Proof</u> Idea is to split A up into sets and define F to be f on a subset of A ~~part of A~~ ~~some sets~~ and g^{-1} on another subset of A on which g^{-1} is defined.

~~Define $a \in A$ to be first generation if~~

Define $A_1 = A \setminus \mathrm{Im}(g) \subset A$. $g(B)$ is a recognised notation for $\mathrm{Im}(g)$. for $B' \subset B$, $g(B') = \{g(b) : b \in B'\}$. $f(A')$ is similar

Define $B_1 = B \setminus \mathrm{Im}(f)$.

For $n \geqslant 1$ $A_{n+1} = g(B_n)$ $B_{n+1} = f(A_n)$

$A_1 \cap A_2 = \phi$ because $A_2 \subset \mathrm{Im}(g)$ and ~~from~~ $A_1 = A \setminus \mathrm{Im}(g)$

Similarly $B_1 \cap B_2 = \phi$.

In fact $A_1 \cap A_n = \phi$ $\forall n \geqslant 2$ because $A_n \subset \mathrm{Im}(g)$ $\forall n \geqslant 2$

and $B_1 \cap B_n = \phi$ $\forall n \geqslant 2$.

Applying g ~~and~~ ~~see that~~ by induction on $A_m \cap A_n = \phi$ $\forall m < n$

$B_m \cap B_n = \phi$ $\forall m < n$.

$A_\infty = A \setminus \bigcup_{n=1}^{\infty} A_n$ $B_\infty = B \setminus \bigcup_{n=1}^{\infty} B_n$

$g: B_n \longrightarrow A_{n+1}$ and $f: A_n \longrightarrow B_{n+1}$ are bijections

So $g^{-1}: A_{n+1} \longrightarrow B_n$ is also a bijection.

$$g : B \longrightarrow \mathrm{Im}(g) = A \setminus A_1 = \bigcup_{n=2}^{\infty} A_n \cup A_\infty$$

So $g : B_\infty \longrightarrow A_\infty$ is a bijection

and $f : A_\infty \longrightarrow B_\infty$ is a bijection

Define $F : A \longrightarrow B$ key

$F = f$ on A_{2n-1} $\forall n \geq 1$, and on A_∞

$= g^{-1}$ on A_{2n} $\forall n \geq 1$.

$F(A_{2n-1}) = B_{2n}$

$F(A_{2n}) = B_{2n-1}$

$F(A_\infty) = B_\infty$

F is a bijection $\square$

<u>Proof of Theorem 1</u>

By the S-B theorem it suffices to find injective maps from $\mathbb{Z}_+$ to $\mathbb{Q}$ and an injective map from $\mathbb{Q}$ to $\mathbb{Z}_+ \subset \mathbb{Z}$.

$g : \mathbb{Z}_+ \longrightarrow \mathbb{Q}$ given by $g(n) = n$ is clearly injective.

So now we need an injective map $f : \mathbb{Q} \longrightarrow \mathbb{Z}_+$

First we find an injective map $f_1 : \mathbb{Q} \longrightarrow \mathbb{Z} \times \mathbb{Z}_+$

by defining $f_1\left(\frac{p}{q}\right) = (p, q)$ if $\gcd(p, q) = 1$, $q \in \mathbb{Z}_+$

$f_1(0) = (0, 1)$

Let $h : \mathbb{Z} \longrightarrow \mathbb{N}$ be injective and define $f_2 : \mathbb{Z} \times \mathbb{Z}_+ \to \mathbb{N} \times \mathbb{Z}$

by $f_2(p, q) = (h(p), q)$

Now we define an injective map $f_3 : \mathbb{N} \times \mathbb{Z}_+ \to \mathbb{Z}_+$

Then $f_3 \circ f_2 \circ f_1 : \mathbb{Q} \to \mathbb{Z}_+$ will be injective.

$$\begin{array}{ccccc}
(0,1) & (0,2) & (0,3) & (0,4) & (0,5) \\
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) \\
(2,1) & (2,2) & (2,3) & & \\
(3,1) & (3,2) & & & \\
(4,1) & & & &
\end{array}$$

Define $f_3(0,1) = 0 \quad f_3(1,1) = 1, \quad f_3(0,2) = 2$

$f_3(2,1) = 3 \quad f_3(\cancel{1},2) = 4 \quad f_3(0,3) = 5 \ldots$

There are n elements (m,p) in the arrays which sum to n. for each $n \geq 1 \quad 1 + \cdots + n-1 = \frac{n(n-1)}{2}$

So $f_3(m,p) = \frac{(m+p)(m+p-1)}{2} + p$

$f_3(0,1) = \frac{1 \times 0}{2} + 1$

$f_3(1,1) = \frac{2 \times \cancel{2}}{2} + 1 = 2 \quad f_3(0,2) = 1 + 2 = 3 \ldots$

f_3 is $\mathbb{N} \times \mathbb{Z}_+ \to \mathbb{Z}_+$ is actually a bijection.

Proof of Theorem 2 In order to prove Theorem 2, it suffices to show there is no surjection from $\mathbb{Z}_+$ to $\mathbb{R}$, because any bijection is surjective.

It suffices to find a subset A of $\mathbb{R}$ such that there is no surjection from $\mathbb{Z}_+$ to A – because if there is a surjection $f_1 : \mathbb{Z}_+ \to \mathbb{R}$ then $f_2 \circ f_1 : \mathbb{Z}_+ \to A$ is a surjection

where
$$f_2(x) = x \text{ if } x \in A$$
$$= a \text{ if } x \notin A$$

for some fixed $a \in A$

Let $A = \{0.a_1a_2\ldots : a_i \in \{0,1\} \forall i\} \subset \mathbb{R}$

$$0.a_1a_2\ldots = \sum_{i=1}^{\infty} \frac{a_i}{10^i}$$

For any n, $\displaystyle\sum_{i=n+1}^{\infty} \frac{a_i}{10^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{10^i} = \frac{\frac{1}{10^{n+1}}}{1-\frac{1}{10}} = \frac{1}{9\times 10^n}$

It follows that if $a_i \in \{0,1\}$ $\forall i$ and $b_i \in \{0,1\}$ $\forall i$ then

$$\sum_{i=1}^{\infty} \frac{a_i}{10^i} = \sum_{i=1}^{\infty} \frac{b_i}{10^i} \iff a_i = b_i \;\forall i$$

Suppose $f_1 : \mathbb{Z}_+ \to A$ is a surjection

Write $\displaystyle f_1(n) = \sum_{i=1}^{\infty} \frac{a_{i,n}}{10^i}$ Define $a_i = 0$ if $a_{i,i} = 1$
$= 1$ if $a_{i,i} = 0$

So $a_{i,i} \neq a_i$. It follows that $\displaystyle\sum_{i=1}^{\infty} \frac{a_i}{10^i} \neq f_1(n)$ for any $n \in \mathbb{Z}_+$

So f_1 is not a surjection, giving the required contradiction.

Examples of the Schröder Bernstein Theorem

(1) $[0,1)$ and $[0,1]$ have the same cardinality.

For ~~$f(x)=$~~ $f:[0,1) \longrightarrow [0,1]$ given by $f(x)=x$ is injective, and $g:[0,1] \longrightarrow [0,1)$ given by $g(x)=\frac{x}{2}$ is injective. The method of proof of Schröder Bernstein Theorem would give $F(x)=f(x)=x$ if $x \in \left(\frac{1}{2^{n+1}},\frac{1}{2^n}\right), n \geq 0$
$F(x) = g^{-1}(x) = 2x$ if $x = \frac{1}{2^n}, n \geq 1$
$F(0)=0$

(2) Also $[0,1]$ and $(0,1)$ have the same cardinality because

$h:[0,1] \longrightarrow (0,1)$ ~~$h=\frac{1}{3}+\frac{x}{3}$~~ given by $h(x)=\frac{1}{3}+\frac{x}{3}$ is injective.

$0 \mapsto 1/3$, $1 \mapsto 2/3$, $1/3 \mapsto 4/9$, $2/3 \mapsto 5/9$ ...

$F(x) = \frac{1+x}{3}$ if $x = \frac{1}{2}\left(1 \pm \frac{1}{3^n}\right)$ $n \in \mathbb{N}$
$F(x) = x$ otherwise

(3) Definition A set A is countable if either A is finite or there is a bijection $f:\mathbb{Z}_+ \longrightarrow A$.
Equivalently A is countable if either $A=\emptyset$ or there is an injective map $f:A \to \mathbb{Z}_+$

Example ~~From sheet 11, if A~~ If A and B are countable then $A \times B$ is countable.

~~e.g. the~~ To see this, there are bijections $f:\mathbb{Z}_+ \longrightarrow A$ and $g:\mathbb{Z}_+ \longrightarrow B$. So $f \times g: \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow A \times B$ is countable, where $(f\times g)(n,m) = (f(n),g(m))$. We know $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable from the proof of Theorem 1.

It follows by induction that ~~A~~ for any $n \in \mathbb{Z}_+$, if A_i is countable for $1 \leq i \leq n$, then $A_1 \times \cdots \times A_n$ is countable.

So $\mathbb{Z}^n$ and $\mathbb{Q}^n$ are countable for all $n \in \mathbb{Z}_+$.