Examples

(1) $A = \{x \in \mathbb{Q} : x < 1\}$ is a Dedekind cut.

$1 \notin A$, so there is no maximal element and conditn 4 is satisfied.

(2) Generalising example 1, fix $a \in \mathbb{Q}$. Then

$A_a = \{x \in \mathbb{Q} : x < a\}$ is a Dedekind cut.

$a \notin A_a$, so there is again no maximal element.

(3) $A = \{x \in \mathbb{Q} : x \leq 1\}$ is not a Dedekind cut, because 1 is a maximal element.

(4) $A = \{x \in \mathbb{Q} : 0 \leq x < 1\}$ is not a Dedekind cut because property 3 is not satisfied

$\frac{1}{2} \in A$ and $0 < \frac{1}{2}$ and yet $0 \notin A$.

(5) $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is not a Dedekind cut, because property 3 is not satisfied.

$0 \in A$ and $-2 < 0$ and yet $-2 \notin A$ because

$(-2)^2 > 2$

(6) $B = \{x \in \mathbb{Q} : x^2 < 2\} \cup \{x \in \mathbb{Q} : x \leq 0\}$

is a Dedekind cut. Properties 1 and 2 are clear.

For 3: if $x \in B$ then either $x \leq 0$ or $0 < x$ and $x^2 < 2$

If $y < x \in B$ and $y \leq 0$ then $y \in B$. If $0 < y < x \in B$ then $y^2 < x^2 < 2 \Rightarrow y \in B$. So property 3 is satisfied.

Now for property 4. We need to show B does not have a maximal element.

If $b \in B$ is maximal then $b \geqslant 1$ because $1 \in B$. We know that $b^2 < 2$. Put $\varepsilon = \dfrac{2-b^2}{3b} \leqslant \frac{1}{3} < 1$

We claim that $b+\varepsilon \in B$ and hence b cannot be ~~maximal~~ maximal.

$$(b+\varepsilon)^2 = b^2 + 2\varepsilon b + \varepsilon^2$$

$$< b^2 + 2 \times \frac{(2-b^2)}{3b} \times b + \varepsilon$$

$$\leq b^2 + 2\times\left(\frac{2-b^2}{3}\right) + \frac{2-b^2}{3}$$

$$= b^2 + 2 - b^2 = 2$$

So $b+\varepsilon \in B$ as required.

So B has no maximal element, and Property 4 holds.

Since there is no rational x with $x^2 = 2$ we also have

$$B = \{x \in \mathbb{Q} : x^2 \leq 2\} \cup \{x \in \mathbb{Q} : x \leq 0\}$$

Def^n A Dedekind cut of the form $\{x \in \mathbb{Q} : x < a\}$ where $a \in \mathbb{Q}$, is called a <u>rational</u> Dedekind cut. These are the examples (2) above. Example (6) is a <u>non-rational</u> Dedekind cut. There are many more examples.

~~Note that for $a, b \in \mathbb{Q}$, $A_a =$~~

Note that for $a, b \in \mathbb{Q}$, $A_a = A_b \Leftrightarrow a = b$.

A simple way to make nonrational Dedekind cuts is to use polynomials with integer coefficients. We want to use Dedekind cuts to *define* real numbers. Therefore the definitions should not themselves use real numbers

<u>Example</u> $f(x) = x^3 + x + 1$ is an increasing function (x^3 and x are increasing. Also, $f'(x) = 3x^2 + 1 > 0$)

$\nexists\, x \in \mathbb{Q}$ such that $f(x) = 0$. To see this, suppose

$$\left(\frac{m}{n}\right)^3 + \frac{m}{n} + 1 = 0 \quad \text{for } m, n \in \mathbb{Z} \setminus \{0\},\ \gcd(m,n) = 1$$

Then $m(m^2 + n^2) = -n^3$. m and $n \neq 0 \Rightarrow n \neq \pm 1$

Suppose p is prime and $p \mid n$. Then $p \mid m \vee p \mid (m^2 + n^2)$

~~In both~~ So $p \mid m$ $\times$.

~~So $\{x \in$~~ Let $A = \{x \in \mathbb{Q} : f(x) < 0\}$

$-1 \in A$ and $0 \notin A$. $f(x) < 0 \wedge y < x \Rightarrow f(y) < 0$

So A satisfies the first 3 properties of a Dedekind cut.

<u>4th property</u>
We can prove directly that A has no maximal element

If $x \in A$ and $-1 \leq x < 0$ and $0 < \varepsilon < 1$ then

$$(x+\varepsilon)^3 + x + \varepsilon + 1 = (x^3 + x + 1) + \underbrace{3\varepsilon x^2}_{\leq 3\varepsilon} + \underbrace{3\varepsilon^2 x}_{<0} + \underbrace{\varepsilon^3}_{<\varepsilon} + \varepsilon$$

$$< x^3 + x + 1 + 5\varepsilon \leq 0 \text{ if } \varepsilon \leq \frac{-f(x)}{5}$$

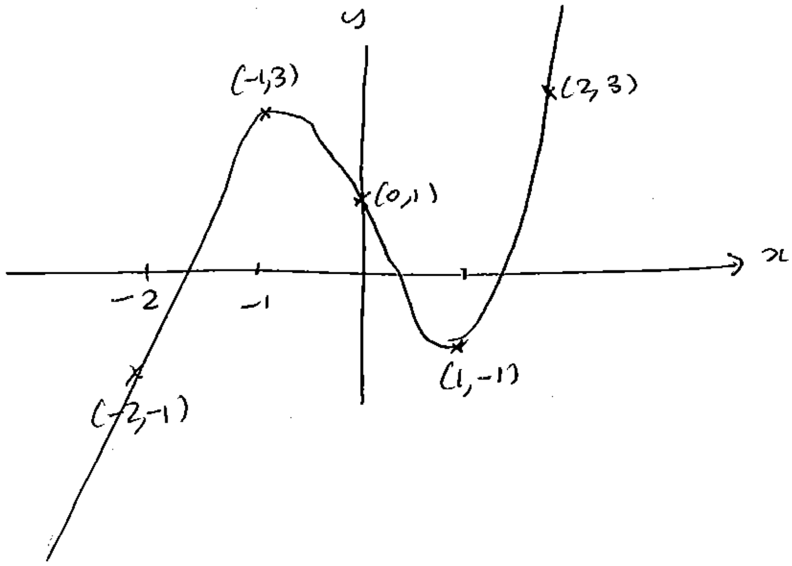
$f(-1) = -1$ so if $x \in A$, $-1 \leq x$, $x + \left(\frac{-f(x)}{5}\right) \in A$ and A has no maximal element

So A is a Dedekind cut. A is not a rational Dedekind cut because if $a \in \mathbb{Q}$ and $A = \{x \in \mathbb{Q} : x < a\}$ then we must have $f(a) = 0$ (by continuity of f – but half of this has been proved directly because the argument just given shows $f(a) \geq 0$) There is no such a, so A is not rational.

<u>Example</u> $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$ $\quad f''(x) = 6x$ $\quad -1$ is local max, 1 is local min

$f(-2) = -1 \quad f(-1) = 3 \quad f(1) = -1 \quad f(2) = 3 \quad f(0) = 1$



A polynomial ~~which~~ with integer coefficients which cannot be factorised as a product of polynomials with integer coefficients has no rational zeros. A "monic" polynomial – one with coefficient ± 1 for the highest term – has no rational zeros if it has no integer zeros.

In any case, this can be proved directly for this example.

$\left(\frac{m}{n}\right)^3 - 3\frac{m}{n} + 1 = 0 \Leftrightarrow m^3 - 3mn = n^3$ Since there are no integer zeros, $n \neq \pm 1$

(77)

So if $m^3-3mn=n^3$, with $\gcd(m,n)=1$, ~~let p pe~~

let p be a prime dividing n.

then $p \mid m^3-3mn$ so $p \mid m$ and $\gcd(m,n)\neq 1$,

a contradiction.

We want to have 3 Dedekind cuts corresponding to "zeros"

of f. $A=\{x\in\mathbb{Q} : f(x)<0\}$ is not a Dedekind cut because,

for example, $1\in A$ and $0\notin A$

However we can form 3 Dedekind cuts corresponding to the

zeros of f:

$A_1=\{x\in\mathbb{Q} : f(x)<0 \wedge x<-1\}=\{x\in\mathbb{Q} : f(x)<0\}\cap\{x\in\mathbb{Q} : x<-1\}$

$A_2=\{x\in\mathbb{Q} : x\leq 0\}\cup\{x\in\mathbb{Q} : f(x)>0 \wedge x<1\}$

$A_3=\{x\in\mathbb{Q} : f(x)<0 \vee x<1\}=\{x\in\mathbb{Q} : f(x)<0\}\cup\{x\in\mathbb{Q} : x<1\}$

For each of these, $-2\in A_i$ and $2\notin A_i$

Property 3 is satisfied for each

e.g. for A_1 if $x\in A_1$ and $y<x$ then $f(y)<f(x)<0$

because f is strictly increasing for $x<-1$.

For A_2 if $x\in A_2$ and $y<x$ then if $y\leq 0$ we have $y\in A_2$

if $y\geq 0$ then $0\leq y<x<1$ and f is strictly decreasing

on $[0,1]$ between y and x so $f(y)>f(x)>0$ and $y\in A_2$

Property 4 in each case can be proved using continuity of the polynomial f - can also be proved from first principles

Using continuity: Suppose $x \in A_i$, and we want to show x is not maximal. If $x \in A_1$ then $f(x) < 0$. If $x \in A_2$ we can assume $x \geqslant 0$ and hence $f(x) > 0$

If $x \in A_3$ we can assume $x \geqslant 1$ and hence $f(x) < 0$

In each case we can find $\delta \in \mathbb{Q}$ with $\delta > 0$ s.t. if $|y - x| < \delta$ then $|f(y) - f(x)| < \frac{|f(x)|}{2}$. In case A_1 we also need $\delta < -x-1$ and in case A_2 we need $\delta < 1-x$

Then $x + \frac{\delta}{2} \in A_i$. So x is not maximal.

Simple Properties of Dedekind Cuts

These 2 lemmas give important properties of Dedekind cuts as regards order

Lemma 1 If A and B are Dedekind cuts then either

$A \subset B$ or $B \subset A$

Proof If $A = B$ both $A \subset B$ and $B \subset A$ are true. So suppose $A \neq B$. Without loss of generality $\exists$ $b \in B$ with $b \notin A$.

By property 3 for A if $a \in A$ then $a < b$, and hence $a \in B$ by property 3 for B. So $A \subset B$ $\square$

Lemma 2 If A is a Dedekind cut and $n \in \mathbb{Z}_+$ then $\exists$ $x \in A$ and $y \notin A$ with $x < y < x + 1/n$.

Proof Choose $x_0 \in A$. Consider $x_0 + \frac{i}{n+1}$ for $i \in \mathbb{N}$, $\exists m \in \mathbb{Z}_+$ s.t. $x + \frac{m}{n+1} \notin A$ otherwise for any $y \in \mathbb{Q}$ we can find m s.t. $y < x_0 + \frac{m}{n+1}$ and $x_0 + \frac{m}{n+1} \in A$ and then $y \in A$. and $\mathbb{Q} \subset A$. So there is a least $m \in \mathbb{Z}_+$ with $x_0 + \frac{m}{n+1} \notin A$

To see this first choose $b_1 \in B \setminus A$. Since b_1 is not maximal we can find $b_2 > b_1$ with $b_2 \in B$

Then $A < A_{b_2} < B$ $\quad b_2 \notin A_{b_2} \Rightarrow A_{b_2} \neq B$

$b_1 \in A_{b_2} \wedge b_1 \notin A \Longrightarrow A_{b_2} \neq A$

So $A < A_{b_2} < B$.

<u>Definition</u> A <u>real number</u> is a Dedekind cut.

An <u>irrational real number</u> is a nonrational Dedekind cut.

The set of real numbers now has an order – given by the order on Dedekind cuts. Order has some of the properties we expect: such as, between any 2 real numbers there is a rational number – because between any 2 ~~of~~ Dedekind cuts, in the ordering, there is a rational number. But what about arithmetic?

<u>Addition</u>

If A and B are Dedekind cuts then we define

$A + B = \{a+b : a \in A, b \in B\}$

We claim that $A+B$ is a Dedekind cut.

1. Clearly $A+B \neq \phi$
2. If $M \notin A$ and $N \notin B$ then $a < M$ $\forall a \in A$, $b < N$ $\forall b \in B$, So $a+b < M+N$ $\forall a \in A$, $b \in B$ and $M+N \notin A+B$.
3. If $x \in \mathbb{Q}$ and $x < a+b$ then $x-a < b$ so $x-a \in B$ and $(x-a)+a = x$ So if $a \in A$ and $b \in B$ and $x \in \mathbb{Q}$ with $x < a+b$, we have $x \in A+B$.

and $x_0 + \frac{m-1}{n+1} \in A$. Put $x = x_0 + \frac{m-1}{n+1}$ and $y = x_0 + \frac{m}{n+1}$

Then $x \in A$, $y \notin A$ and $x < y < x + \frac{1}{n}$ $\square$.

Order on Dedekind cut's

Defn For Dedekind cuts A and B we define $A < B$ if $A \subset B$ and $A \neq B$.

From Lemma 1 we then get the following important property of order.

For any 2 Dedekind cuts A and B, exactly one of the following holds:

$$A < B \qquad A = B \qquad B < A.$$

For rational Dedekind cuts $A_a \doteq \{x \in \mathbb{Q} : x < a\}$ and $A_b = \{x \in \mathbb{Q} : x < b\}$, for $a, b \in \mathbb{Q}$, we have

$$A_a < A_b \iff a < b.$$

So order on rational Dedekind cuts coincides with the order on the corresponding rational numbers.

We also have the following ~~For any 2~~

For any Dedekind cuts A ~~and~~ $< B$ there is $b \in \mathbb{Q}$ ~~corresponding~~ corresponding to rational Dedekind cut A_b such that

$$A < A_b < B$$

~~Simply chose $b \in B \setminus A$ and b' > b and then $A \subset A_{b'} \subset B$ ~~ ~~and $b \notin A_b \Rightarrow A_b \neq B$~~

~~$b \notin A$~~

4. To show no maximal element: any element of $A+B$ is of the form $a+b$ with $a \in A$ and $b \in B$. a is not maximal in A. So $\exists\, a' \in A$ with $a' > a$.

Then $a'+b > a+b$ and $a'+b \in A+B$.

So no element of $A+B$ is maximal.

~~Order on Dedekind cuts~~

~~has the property that exactly one of the following holds~~

~~$A < B \quad A = B \quad B < A$.~~

Addition of Dedekind cuts is associative, commutative and distributive. If $a \in \mathbb{Q}$ we can write a instead of $A_a = \{x \in \mathbb{Q} : x < a\}$.

If B is any Dedekind cut then

$B + A_a = \{x+a : x \in B\}$ because if $x \in B$

$\exists\, x' > x$ with $x' \in B$ and then $x + a = x' + \underbrace{a + (x - x')}_{\in A}$

In particular $B + 0 = B = 0 + B$.

Addition also has the usual properties associated to order:

e.g. if A, B, C are Dedekind cuts, and $B < C$ then $A + B < A + C$.

The Dedekind cut $-A$

We define $-A = \mathbb{Q} \setminus \{-x : x \in A\}$ if A is a nonrational Dedekind cut

$-A = \mathbb{Q} \setminus \{-x : x \in A\} \setminus \{-a\}$ if A is a rational Dedekind cut $A = A_a$ with $a \in \mathbb{Q}$

Then $-A$ is a Dedekind cut

If A is nonrational then $y \in -A \iff (y < -x \ \forall x \in A \wedge y \in \mathbb{Q})$

If A is rational with $A = A_a$ for $a \in \mathbb{Q}$ then

$y \in -A \iff (y < -x \ \forall x \in A \wedge y \neq -a \wedge y \in \mathbb{Q})$

$\iff y < -a \wedge y \in \mathbb{Q}$

So $-A_a = A_{(-a)}$

Properties 1, 2, 3 are clearly satisfied. Also Property 4 is satisfied because if $-A$ has a maximal element b and A is nonrational

then $b \in \mathbb{Q}$ and $-A = \{x \in \mathbb{Q} : x \leq b\}$

That means $\{-x : x \in A\} = \{y \in \mathbb{Q} : y > b\}$

$-x > b \iff x < -b$

and $A = \{x : x < -b\}$ is rational. X

If $A = A_a$ is rational then $-A = A_{-a}$ so $-A$ is a Dedekind cut.

An important result about $-A$ is

Theorem If A and B are Dedekind cuts, then $A + B = 0$ $\iff B = -A$

Proof First we show $A + (-A) = 0$. Since $y < -x \ \forall y \in -A$ and $x \in A$ we have $A + -A \leq 0$. Now we show $A + (-A) = 0$. By Lemma 2 given $n \in \mathbb{Z}_+$ $\exists a \in A$ and $c \notin A$ with $a < c < a + \frac{1}{n}$

$c > a' \ \forall a' \in A$ So $-c < -a' \ \forall a' \in A$

So $-c \in -A$ $a - c > -1/n$.

So $a - c \in A + (-A) \Rightarrow -\frac{1}{n} \in A + (-A)$

This is true $\forall n \in \mathbb{Z}_+$ If $y \in \mathbb{Q}$ and $y < 0$

then $\exists n \in \mathbb{Z}_+$ s.t. $y < -1/n$.

So $y \in A + (-A)$ $\forall y < 0$ and $A + (-A) = O (= A_0)$.

Now suppose $A + B = O$. Then $a + b < 0 \ \forall b \in B$.

$a + b < 0 \ \forall a \in A, b \in B$ So $b < -a \ \forall a \in A, b \in B$.

So $B \leq -A$ (If A is rational, we need to use the fact that B is a Dedekind cut, and hence does not have a rational element.)

If $B \neq -A$ then $c' < c \in -A$ with $c', c \notin B$.

$b < c' < c \quad \forall b \in B$.

$c \in -A \Rightarrow c + a < 0 \quad \forall a \in A$

$b + a < \cancel{c+a} + c' + a \quad \forall b \in B, a \in A$

$b + a < c + a + (c' - c) < c' - c \quad \forall b \in B, a \in A$

So $A + B \leq A_{c'-c}$ $\times$ $\square$.

<u>Corollary</u> For all Dedekind cuts A and B

$-(A + B) = (-A) + (-B)$

Proof $(A + B) + ((-A) + (-B)) = (A + (-A)) + (B + (-B)) = O$

So $-(A+B) = (-A) + (-B)$ from the theorem.

Multiplication

For Dedekind cuts A, B with $A \geq 0$ and $B \geq 0$ we define

$$A.B = \{x \in \mathbb{Q} : x \leq 0\} \cup \{ab : a \in A, b \in B \text{ with } a \geq 0, b \geq 0\}$$

This is a Dedekind cut. If $a < M$ and $b < N$ $\forall a \in A, b \in B$ then $MN \notin A.B$

If $x = a.b \in A.B$ and $y < x$, if $y \leq 0$ then $y \in A.B$. If $y > 0$ then $0 < y < x = a.b$ with $a > 0, b > 0$

$$\Rightarrow \frac{y}{b} < a \Rightarrow \frac{y}{b} \in A \Rightarrow \frac{y}{b}.b = y \in A.B$$

Also if $a.b$ is maximal in $A.B$ then a is maximal in A and b is maximal in B giving a contradiction

For ~~general~~ other Dedekind cuts we define

$$\begin{aligned} A.B &= -((-A).B) \text{ if } A < 0, B \geq 0 \\ &= -(A.(-B)) \text{ if } A \geq 0, B < 0 \\ &= (-A).(-B) \text{ if } A < 0, B < 0 \end{aligned}$$

$$A_a . A_b = A_{ab} \quad \forall a, b \in \mathbb{Q}$$

All the usual rules apply e.g. $0.A = 0 \ \forall A$
$1.A = A \ \forall A$.

Multiplicative Inverse

If $A > 0$ we define $A^{-1} = \mathbb{Q} \setminus \{x^{-1} : x \in A, x > 0\}$ if A is irrational

$A^{-1} = \mathbb{Q} \setminus (\{x^{-1} : x \in A, x > 0\} \cup \{a^{-1}\})$ if $A = A_a$, $a \in \mathbb{Q}$

If $A < 0$ we define $A^{-1} = -(-A)^{-1}$

Examples An important motivation of ~~the construction~~ of Dedekind cuts is to find a number whose square is 2.

If $A = \{x \in \mathbb{Q} : x \leq 0 \vee x^2 < 2\}$ then A is a Dedekind cut with $A^2 = 2$. To see this:

Clearly $x \cdot y < 2$ $\forall\ 0 < x, y$ with $x^2 < 2$ and $y^2 < 2$

So $A^2 \leq 2$. To show $A^2 = 2$, given $n \in \mathbb{Z}_+$ we can find $0 < x \in A$ with $x + \frac{1}{n} \notin A$

So $x^2 < 2$ and $(x + 1/n)^2 > 2$ $\quad (x+\frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} > 2$

So $\cancel{\frac{2x}{n}}$ $\cancel{x^2}$ $x^2 > 2 - \frac{2x}{n} - \frac{1}{n^2}$

$x < 2$ so $x^2 > 2 - \frac{4}{n} - \frac{1}{n^2} \geq 2 - \frac{5}{n}$

So $2 - \frac{5}{n} \leq A^2$ $\forall n \in \mathbb{Z}_+$

So $A^2 = 2$.

Similarly if $A = \{x \in \mathbb{Q} : x \leq 0 \vee x^2 < p\}$ for any $p \in \mathbb{Q}$ with $p > 0$ then $A^2 = p$

This can be shown in exactly the same way $A^2 \leq p$

Choose $x \in A$ with $x + \frac{1}{n} \notin A$ $\quad x^2 < p < x^2 + \frac{2x}{n} + \frac{1}{n^2}$

$x^2 > p - \frac{2x}{n} - \frac{1}{n^2}$ $\quad$ $x \leq p+1$ so $x^2 > p - \frac{2p+2}{n} - \frac{1}{n^2}$

$p - \frac{2p+3}{n^2} \leq A^2\ \forall n \in \mathbb{Z}_+$. So $A^2 = p$

Similarly (but a bit harder) we can show that if $A = \{x \in \mathbb{Q} : x^3 + x + 1 < 0\}$ then $A^3 + A + 1 = 0$ . . .

Properties of Real numbers

$\mathbb{R}$ is the set of real numbers

We now consider the elements of $\mathbb{R}$ as numbers and consider their properties. $\mathbb{Q} \subset \mathbb{R}$. So properties should be consistent with those of $\mathbb{Q}$

Properties of arithmetic

Addition, multiplication, subtraction are defined for any pair of real numbers. ~~Division $x \div y = \frac{x}{y}$ is defined~~
Special real numbers 0, 1 are defined. Division $x \div y = \frac{x}{y}$ is defined for any $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$.

Addition and multiplication are both associative and commutative

$(x+y)+z = x+(y+z) \ \forall x, y, z \in \mathbb{R} \qquad x+y = y+x \ \forall x, y \in \mathbb{R}$

$(x \cdot y) \cdot z = x \cdot (y \cdot z) \ \forall x, y, z \in \mathbb{R} \qquad x \cdot y = y \cdot x \ \forall x, y \in \mathbb{R}$

In addition the distributive law holds

$$x \cdot (y+z) = (x \cdot y) + (x \cdot z) \quad \forall x, y, z \in \mathbb{R}$$

Special properties of 0, 1:

$$x + 0 = 0 + x = x \ \forall x \in \mathbb{R}$$

$$x \cdot 1 = 1 \cdot x = x \ \forall x \in \mathbb{R}$$

$$\forall x \in \mathbb{R}, \ \exists -x \in \mathbb{R} \text{ s.t. } x + (-x) = (-x) + x = 0$$

$$\forall x \in \mathbb{R} \setminus \{0\}, \ \exists \tfrac{1}{x} \in \mathbb{R} \text{ s.t. } x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$$

Other properties can be deduced e.g. $x + y = 0 \Leftrightarrow y = -x$

$$0 \cdot x = 0 \ \forall x \in \mathbb{R}.$$

Subtraction is then defined by $x - y = x + (-y)$

Division is defined by $x \div y = x \cdot \frac{1}{y}$ for $y \neq 0$

There are also properties of order, and properties relating order and arithmetic.

Order properties

$\forall x, y \in \mathbb{R}$, exactly one of the following holds

$$x < y \qquad x = y \qquad y < x$$

Transitivity $(x < y \wedge y < z) \implies x < z \quad \forall x, y, z \in \mathbb{R}$

Properties relating order and arithmetic

$$\forall x, y, z \in \mathbb{R}, \quad y < z \implies x + y < x + z$$

$$\forall x, y, z \in \mathbb{R} \ (y < z \wedge 0 < x) \implies xy < xz$$

But $\mathbb{Q}$ has all these properties. We know that $\mathbb{R}$ has more elements than $\mathbb{Q}$. What really distinguishes $\mathbb{R}$ from $\mathbb{Q}$?

Let's leave this open for the moment.

Sequences

Def^n A sequence in X is a function $f: \mathbb{N} \to X$, or $f: \mathbb{Z}_+ \to X$ or (occasionally) $f: \{n \in \mathbb{N} : n \geq k\} \to X$ for some $k \in \mathbb{N}$. We write $f(n) = x_n$, and then the sequence is $\{x_n : n \geq k\}$ (usually with $k = 0$ or 1)

We will be especially interested in sequences in $\mathbb{R}$. Often, they will be sequences in $\mathbb{Q}$, but we will still think of them as sequences in $\mathbb{R}$.

Examples ① $x_n = \frac{1}{n}$ $\quad n \geq 1$ $\quad x_1 = 1,\ x_2 = \frac{1}{2},\ x_3 = \frac{1}{3},\ x_4 = \frac{1}{4} \ldots$

② $x_n = 2^n \quad n \geq 0$

③ $x_n = \frac{n}{n+1} \quad n \geq 0$

④ $x_n = 1 + \frac{1}{2} \ldots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \quad n \geq 1$

⑤ $x_n = 1 + \frac{1}{2^2} \cdots + \frac{1}{n^2} = \sum_{k=1}^{n} \frac{1}{k^2}$

⑥ x_n defined inductively by $x_0 = 2$ and, for $n \geq 0$

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \qquad x_1 = \frac{3}{2} \quad x_2 = \frac{17}{12} \quad x_3 = \frac{577}{408} \ldots$$

⑦ $a_n \in \mathbb{Z}_+ \quad n \geq 1,\ n \in \mathbb{Z}_+$

$$x_1 = \frac{1}{a_1} \qquad x_2 = \cfrac{1}{a_1 + \cfrac{1}{a_2}} \qquad x_3 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}} \quad \ldots \quad x_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{}{+\cfrac{1}{a_n}}}}$$

x

Equivalently, $x_n = \frac{p_n}{q_n}$ where $p_1 = 1\ q_1 = a_1 \quad p_0 = 0\ q_0 = 1 \quad p_{-1} = 1\ q_{-1} = 0$

$$p_n = p_{n-2} + a_n p_{n-1} \qquad q_n = q_{n-2} + a_n q_{n-1}$$

Decreasing and increasing sequences

Defⁿ A sequence $\{x_n\}_{n \geq k}$ of real numbers is increasing if $x_n \leq x_{n+1}$ $\forall n$.

It is decreasing if $x_{n+1} \leq x_n$ $\forall n$.

Examples

(1) $x_n = 1/n$ $\forall n \geq 1$ is decreasing because $\frac{1}{n+1} \leq \frac{1}{n}$ $\forall n \geq 1$

(2) $x_n = 2^n$ is increasing because $2^n \leq 2^{n+1}$ $\forall n$.

(3) $x_n = \frac{n}{n+1} = 1 - \frac{1}{n+1}$ is increasing because $1 - \frac{1}{n} < 1 - \frac{1}{n+1}$ $\forall n$

(4) $x_n = 1 + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$ is increasing because

$x_{n+1} - x_n = \frac{1}{n+1} > 0$ $\forall n$.

(5) $x_n = 1 + \cdots + \frac{1}{n^2} = \sum_{k=1}^{n} \frac{1}{k^2}$ is increasing because

$x_{n+1} - x_n = \frac{1}{(n+1)^2} > 0$ $\forall n$

(6) $x_{n+1} - x_n = \frac{x_n}{2} - x_n + \frac{1}{x_n} = \frac{2 - x_n^2}{x_n} = \frac{-(x_{n-1}^2 - 2)^2}{4x_{n-1}^2 \times x_n}$ $\forall n \geq 1$

$x_n > 0$ $\forall n$ by induction. So $x_n^2 \geq 2$ $\forall n$ and

$x_{n+1} - x_n < 0$ $\forall n \geq 1$ also $x_1 - x_0 = -1/4 < 0$

So $\{x_n\}$ is decreasing

(7) This is a general class of examples, which is neither increasing nor decreasing. First, to see some examples

Example $a_n = 3 \; \forall n$

$$x_1 = \frac{1}{3} \qquad x_2 = \frac{1}{3+\frac{1}{3}} = \frac{3}{10} \qquad x_2 = \frac{1}{3+\frac{1}{3+\frac{1}{3}}}$$

$$= \frac{1}{3+\frac{3}{10}} = \frac{10}{33}$$. In general in this example $x_{n+1} = \frac{1}{3+x_n}$

$$x_{n+2} - x_{n+1} = \frac{1}{3+x_{n+1}} - \frac{1}{3+x_n} = \frac{x_n - x_{n+1}}{(3+x_{n+1})(3+x_n)}$$

This suggests $x_{n+1} - x_{n+2}$ has opposite sign to $x_n - x_{n+1}$.

This is true in general.

If $x_n = \frac{p_n}{q_n}$ and $x_{n+1} = \frac{p_{n+1}}{q_{n+1}}$ then

$$x_{n+1} - x_n = \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{p_{n+1}q_n - p_n q_{n+1}}{q_n q_{n+1}}$$

It is an exercise on Sheet 10 to show that

$$p_{n+1}q_n - p_n q_{n+1} = (-1)^n = p_{n-1}q_n - p_n q_{n-1}$$

So $p_{-1}q_0 - p_0 q_{-1} = 1 - 0 = 1$. So true for $n=0$.

Bounded Sequences

A sequence of real numbers $\{x_n\}$ is bounded above if $\exists M$ such that $x_n \leq M \; \forall n$ and bounded below if $\exists L$ s.t. $L \leq x_n \; \forall n$ and bounded if $\exists M$ s.t. $|x_n| \leq M$ $\forall n$, that is $-M \leq x_n \leq M \; \forall n$

Examples

(1) $\{\frac{1}{n}\}(n\geq 1)$ is bounded because $0 \leq \frac{1}{n} \leq 1 \ \forall n \geq 1$

(2) $\{2^n\}(n \geq 0)$ is not bounded above because $2^n \geq n \ \forall n \geq 0$ and $\{n\}$ is not bounded. However, it is bounded below by 1 because $2^n \geq 1 \ \forall n \geq 0$

(3) $\{\frac{n}{n+1}\}$ is bounded, because $0 \leq \frac{n}{n+1} \leq 1 \ \forall n \geq 0$

(4) $\{\sum_{k=1}^{n} \frac{1}{k}\}$ is bounded below by 1, $\forall n \geq 1$ but, perhaps surprisingly, is not bounded above. This is (or will be) shown in MATH 101/2

(5) In contrast, $\{\sum_{k=1}^{n} \frac{1}{k^2}\}$ is bounded below by 1 and above by 2

(6) If $x_0 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ then $\{x_n\}$ is bounded above by 2 and below by 0 as it is a decreasing sequence of positive numbers.

(7) In general, any decreasing sequence $\{x_n : n \geq k\}$ is bounded above by x_k and any increasing sequence $\{x_n : n \geq k\}$ is bounded below by x_k

(7) $a_n \in \mathbb{Z}_+ \ \forall n \geq 1$ $x_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n}}} = \frac{p_n}{q_n}$

$\{x_{2n}\}$ is an increasing sequence and $\{x_{2n+1}\}$ is decreasing with $x_{2n} \leq x_{2n-1} \ \forall n \geq 1$

Then $\{x_n\}$ is bounded above by x_1 and below by x_2.

Limits

Defn A sequence $\{x_n\}$ has limit l, or converges to l, written $\lim_{n\to\infty} x_n = l$, if given $\varepsilon > 0$ $\exists N$ s.t. $|x_n - l| < \varepsilon$ $\forall n \geq N$.

Examples

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

2. $\{2^n ; n \geq 0\}$ does not have a limit (although we write $\lim_{n\to\infty} 2^n = \infty$)

3. $\lim_{n\to\infty} \frac{n}{n+1} = 1$

4. If $x_n = \sum_{k=1}^{n} \frac{1}{k}$ then $\{x_n\}$ does not have a limit (although we can write $\lim_{n\to\infty} x_n = \infty$)

5. If $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ then $\lim_{n\to\infty} x_n = \frac{\pi^2}{6}$

See next year. ~~At~~ We will see shortly that a limit does exist.

6. ~~for~~ We will see shortly that ~~A~~ if $x_0 = 2$ $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ then $\{x_n\}$ has a limit. ~~l~~. If it does have a limit l then $l = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(\frac{x_n}{2} + \frac{1}{x_n}\right) = \frac{l}{2} + \frac{1}{l}$ and $l^2 = 2$

7. We will see shortly that $x_n = \cfrac{1}{a_1 + \cdots + \cfrac{1}{a_n}}$ has a limit $\forall \{a_n\}$ with $a_n \in \mathbb{Z}_+$ $\forall n$. In the example $a_n = 3$ $\forall n$ for which $x_{n+1} = \frac{1}{3 + x_n}$ the limit l satisfies $l = \frac{1}{3+l}$

<u>Theorem</u> (See also MATH 101) If a sequence $\{x_n\}_{n\geq k}$ has a limit, then $\{x_n\}_{n\geq k}$ is bdded.

<u>Proof</u> $\exists N$ s.t. $|x_n - l| \leq 1 \quad \forall n \geq N$

Then $|x_n| \leq |l| + |x_n - l| \leq |l| + 1 \quad \forall n \geq N$

Define $M = \operatorname{Max}(\{|x_i| : k \leq i \leq N\} \cup \{|l|+1\})$

Then $|x_n| \leq M \quad \forall n \geq k$ $\square$

The <u>converse</u> of this theorem is not true. If a sequence $\{x_n\}$ is bounded, it might not have a limit. e.g. $\{(-1)^n\}_{n\geq 0}$ does not have a limit.

Nevertheless, there is a very important property of increasing/decreasing sequences regarding limits.

<u>Completeness Axiom</u> If $\{x_n\}_{n\geq k}$ is an increasing sequence in $\mathbb{R}$ which is bounded above, then $\lim_{n\to\infty} x_n$ exists (in $\mathbb{R}$)

Similarly, if $\{x_n\}$ is a decreasing sequence which is bounded below, then $\lim_{n\to\infty} x_n$ exists (in $\mathbb{R}$)

The Completeness Axiom is the property of the set $\mathbb{R}$ of real numbers which distinguishes it from the set $\mathbb{Q}$ of rational numbers.

<u>Proof of the Completeness Axiom</u>

This proof involves identifying real numbers with Dedekind cuts again. So let $\{A_n\}_{n\geq 1}$ be an <u>increasing</u> sequence of Dedekind cuts which is bounded above. <u>Increasing</u> means $A_n \subset A_{n+1}\ \forall n$. <u>Bounded above</u> means $A_n \subset \{x \in \mathbb{Q} : x < M\}\ \forall n$, that is $a < M \quad \forall a \in A_n, \forall n \geq 1$.

Then $A = \bigcup_{n=1}^{\infty} A_n$ is a Dedekind cut.

$a < M \ \forall a \in \bigcup_{n=1}^{\infty} A_n = \{a' : a' \in A_n \text{ for some } n\}$

$A_n \subset A \ \forall n$ So $A \neq \emptyset$.

$x \in A \wedge y < x \Rightarrow (x \in A_n \text{ for some } n) \wedge y < x \Rightarrow y \in A_n \Rightarrow y \in A$ (Prop 3)

If $a \in A$ is maximal then $a \in A_n$ for some n.

$a' \leq a \ \forall a' \in A_n \Rightarrow a$ maximal in A_n X. So A does not have a maximal element.

So A is a Dedekind cut.

To see $\lim_{n\to\infty} A_n = A$: given $k \in \mathbb{Z}_+$ choose $x \in A$ with $x + 1/k \notin A$. Then $x \in A_n$ for some n.

Then $\{y : y < x\} \subset A_m \subset \{y : y < x + 1/k\}$ for all $m \geq n$.

This means $\lim_{n\to\infty} A_n = A$

Applications

Example (5) $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$ Let's accept $x_n \leq 2 \ \forall n$.

Then $\{x_n\}$ is an increasing sequence and bounded above.

So $\lim_{n\to\infty} x_n$ exists

Example (6) $x_0 = 2 \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$.

$\{x_n\}$ is decreasing and bounded below by 0. So $\lim_{n\to\infty} x_n$ exists. If the limit is l then $l = \frac{l}{2} + \frac{1}{l}$

So $l^2 = 2$ and $l = \sqrt{2}$.

Example (7) x_{2n} is increasing and x_{2n+1} decreasing with $x_{2n} \leq x_{2n+1}$

So $\lim_{n\to\infty} x_{2n}$ exists and $\lim_{n\to\infty} x_{2n+1}$ exists. In addition we can show $\lim_{n\to\infty} x_{2n+1} - x_{2n} = 0$ So $\lim_{n\to\infty} x_n$ exists. In the specific example ~~...~~ $a_n = 3 \ \forall n$ $l = \frac{1}{\sqrt{3}+1}$

Continued Fractions and Decimals

Any irrational real number between 0 and 1 is

$\lim_{n \to \infty} x_n$ for some sequence $\{a_n\}_{n \geq 1}$ with $a_n \in \mathbb{Z}_+ \; \forall n$

and $x_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$

Any rational real number between 0 and 1 is of the form

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 \cdots + \cfrac{1}{a_n}}} \quad \text{for some } n \text{ and } a_i \in \mathbb{Z}_+, \; 1 \leq i \leq n.$$

If the sequence $\{a_n\}$ repeats an initial segment infinitely often, that is, if $\exists \, k \geq 1$ such that $a_{n+k} = a_n \; \forall n \geq 1$

then $x = \lim x_n$ satisfies a quadratic equation of the form $ax^2 + bx + c = 0$

This is also true if $\{a_n\}$ is eventually periodic

that is, there are $k \in \mathbb{Z}_+$ and $N \in \mathbb{Z}_+$ such that

$a_{n+k} = a_n \; \forall n \geq N.$

Decimals

Every real number has an infinite decimal expansion of the form

$x = a_0 . a_1 a_2 a_3 \ldots$

where $a_0 \in \mathbb{Z}$ and $a_i \in \{ n \in \mathbb{N} : 0 \le n \le 9 \}$

Where this means

$$x = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{a_i}{10^i} \quad \text{if } a_0 \geq 0$$

$$x = \lim_{n \to \infty} \left(a_0 - \sum_{i=1}^{n} \frac{a_i}{10^i} \right) \text{ if } a_0 < 0$$

The sequence (a_n) is eventually periodic $\iff$ x is rational

That is $\exists N, k \in \mathbb{Z}_+$ s.t. $a_{n+k} = a_n \; \forall n \geq N \iff x$ is rational.

Example $\frac{1}{13}$

$$\begin{array}{r|l} & .7692376923\ldots \\ 13 & 1.0000000 \\ & \underline{91} \\ & 90 \\ & \underline{78} \\ & 120 \\ & \underline{117} \\ & 30 \\ & \underline{26} \\ & 40 \\ & \underline{39} \\ & 1 \end{array}$$

$0.413413 \ldots$

$$= \frac{4}{10} + \frac{1}{100} + \frac{3}{1000} \left(1 + \frac{1}{1000} + \cdots \right)$$

$$= \frac{413}{1000} \times \frac{1}{1 - \frac{1}{1000}} = \frac{413}{999}$$