What is number?

There appear to be many more rational numbers than integers because of this "in between" property. So is every number rational? It depends what a number is. One would like to be able to associate a number to any length.

The Greeks thought of pure number as being the ratio, of two lengths, or the proportion of length.

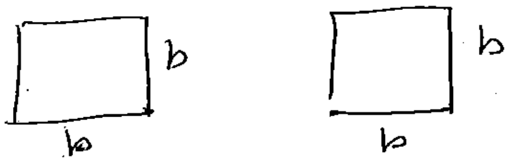
If one rod (for example) has length a and the second has length b, then the first rod has proportion $\frac{a}{b}$ of the second

If $\frac{a}{b} = 2$, then the first rod can be cut into 2 rods of equal length to the second.

If $\frac{a}{b} = \frac{1}{2}$ then the second rod can be cut into 2 rods of length equal to the first.

If $\frac{a}{b} = \frac{3}{2}$ then the first rod can be cut into 3, and the second into 2, so that all 5 bits have equal length.

How can other numbers arise? Suppose you have 2 squares of equal sizes, and you want to make another square out of these. Can it be done?

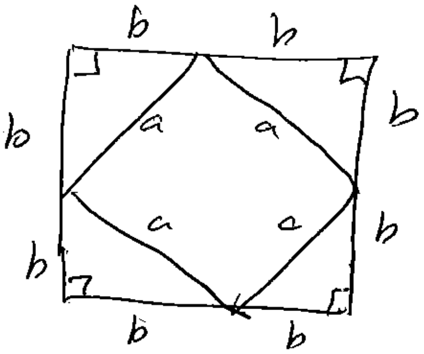


If the new square has side length a then we need $a^2 = 2b^2$ and hence $\left(\frac{a}{b}\right)^2 = 2$

If $\frac{a}{b} = \frac{m}{n}$ with $m, n \in \mathbb{Z}_+$, then we can simply take m^2 squares of side length $\frac{b}{n}$, and make a square of $m \times \frac{b}{n} = a$ sidelength with $a^2 = 2b^2$ for the area. Of course this is impossible

This problem arose in Indian mathematics as the Indian altar problem How to double the size of a square altar, using stone from two existing altars?

The solution found was to use approximation - as we shall see in a bit. Of course, another way would be to split diagonally



From an altar of side length $2b$ we can make 2 altars this way of side length a, with $b^2 + b^2 = 2b^2 = a^2$

$$4b^2 = 2a^2 \quad ; \quad a^2 = 2b^2$$

__Theorem__ If $x^2 = 2$ then $x \notin \mathbb{Q}$, that is, there do not exist $m, n \in \mathbb{Z}_+$ with $\left(\frac{m}{n}\right)^2 = 2$

Proof Suppose $m, n \in \mathbb{Z}_+$ do exist with $\left(\frac{m}{n}\right)^2 = 2$

Take the least possible n ~~with~~ $\in \mathbb{Z}_+$ for which there exists $m \in \mathbb{Z}_+$ with $\left(\frac{m}{n}\right)^2 = 2$ Then $\gcd(m,n) = 1$. For if $k = \gcd(m,n)$ then we have $m_1 k = m$ and $n_1 k = n$ for $m_1, n_1 \in \mathbb{Z}_+$. Then $\left(\frac{m}{n}\right)^2 = 2 \Rightarrow m^2 = 2n^2$ $\Rightarrow m_1^2 k^2 = 2n_1^2 k^2 \Rightarrow m_1^2 = 2n_1^2 \Rightarrow n_1 \geq n$ because n is the smallest possible integer $\in \mathbb{Z}_+$ for which this is true.

So $n_1 k = n \Rightarrow k = 1$.

Now $n^2 = 2m^2 \Rightarrow n^2$ is even $\Rightarrow n$ is even $\Rightarrow n = 2n_1$, some $n_1 \in \mathbb{Z}_+$ Then $4n_1^2 = 2m^2 \Rightarrow 2n_1^2 = m^2 \Rightarrow m^2$ is even $\Rightarrow m$ is even.

So $\gcd(m,n) \geq 2$. $\times$. So m and n do not exist $\square$

__Theorem__ If $p \in \mathbb{Z}_+$ and p is not the square of any integer, then there is no $x \in \mathbb{Q}$ with $x^2 = p$.

Proof Write $p = \prod_{i=1}^{r} p_i^{n_i}$ with p_i distinct primes and $n_i \in \mathbb{Z}_+$

At least one n_i is odd, because otherwise p is the square of an integer. Renumbering, assume n_1 is odd, $n_1 = 2k+1$ for some $k \in \mathbb{N}$

Then $p = p_1^{2k+1} q$ where $q \in \mathbb{Z}_+$ and $\gcd(p_1, q) = 1$

Suppose $x = \frac{m}{n} \in \mathbb{Q}$ with $m, n \in \mathbb{Z}_+$

Write $m = p_1^{s_1} a$ and $n = p_1^{t_1} b$ where $\gcd(a, p_1) = \gcd(b, p_1) = 1$

Then $m^2 = pn^2 \Rightarrow p_1^{2s_1} a^2 = p_1^{2k+1} q \, p_1^{2t_1} b^2 = p_1^{2k+2t_1+1} q b^2$

$$\gcd(p_1, a^2) = \gcd(p_1, qb^2) = 1$$

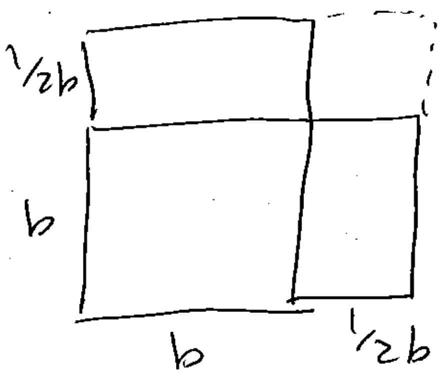
Each of a^2, qb^2 can be written as a product of prime powers <u>not including p_1</u>. This is impossible because the representation of ~~a~~ positive integer as a product of prime powers <u>is unique</u> and $m^2 = n^2 p$ has been written in two ways as a product of prime powers, once with an <u>odd</u> power of p_1 and once with an <u>even</u> power. $\square$

Back to ~~the~~

<u>The Indian altar problem</u>

How <u>do</u> we double the ~~size~~ area of the altar, keeping it square?

We could ~~do~~ add half the length again to the 2 adjacent sides



But there is a missing square.

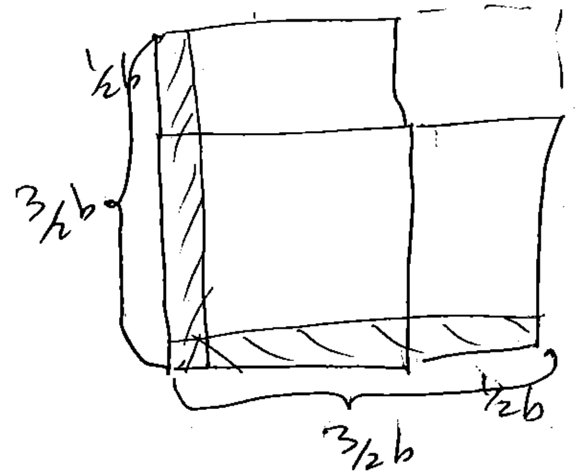
The area <u>with</u> the missing square would be

$$(b+\tfrac{1}{2}b)^2 = b^2(1+\tfrac{1}{2})^2 = b^2(1+1+\tfrac{1}{4})$$
$$= 2b^2 + \tfrac{1}{4}b^2$$

The missing square has area $\frac{1}{4}b^2$

We can try "shaving off" rectangles on the 2 complete sides and filling in the missing square of area $\frac{1}{4}b^2$

(70.1)



Take each of the ~~red~~ shaded or rectangles to have area $\frac{1}{8}b^2$

They each have a long side of length $\frac{3}{2}b$. So ~~take~~ short sides of length

$$\frac{1}{8}b^2 \div \frac{3}{2}b = \frac{1}{8}b^2 \times \frac{2}{3b}$$

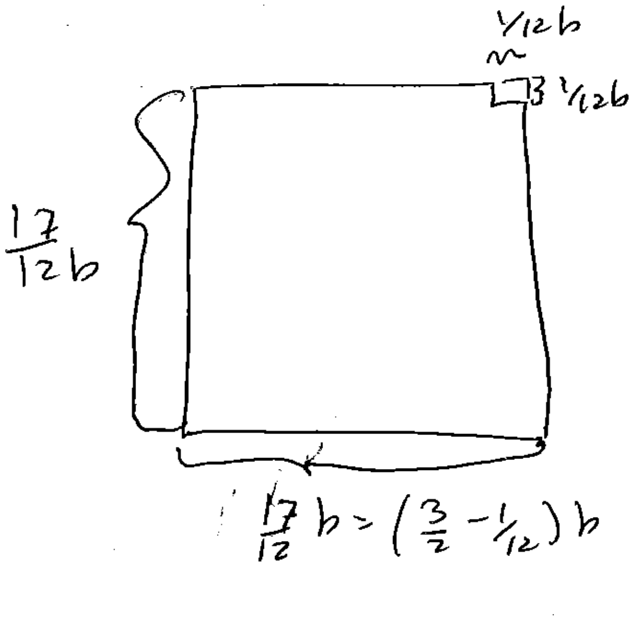
$$= \frac{1}{12}b$$

Each rectangle can be divided into squares of side length $\frac{1}{12}b$ and they can be used to fill up the missing square of ~~side~~ area ~~length~~ $\frac{1}{4}b^2$ and side length $\frac{1}{2}b$.

There are 18 squares in each rectangle but they overlap in one square of side length $\frac{1}{12}b$

$$\left(\frac{3}{2}b - \frac{1}{12}b\right)^2 = b^2\left(\frac{3}{2} - \frac{1}{12}\right)^2 = b^2\,\frac{289}{144} = b^2\left(2 + \frac{1}{144}\right)$$

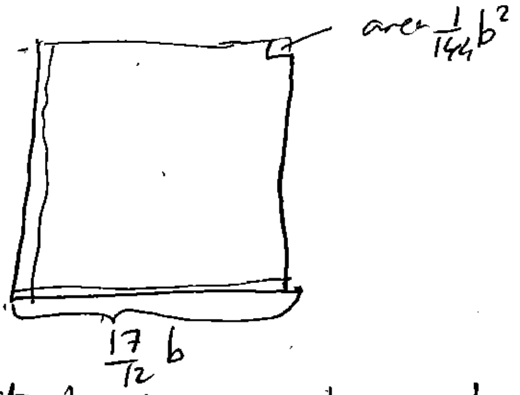
There is a missing square of side length $\frac{1}{12}b$ and area $\frac{1}{144}b^2$



$\frac{17}{12}b = \left(\frac{3}{2} - \frac{1}{12}\right)b$

(70.2)

Try to fill it in by shaving off 2 rectangles from bottom + left-hand sides



Take each shaved off rectangle of area $\frac{1}{2\times 144} = \frac{1}{288}b^2$

sidelengths $\frac{17}{12}b$ and c

So $\frac{17}{12}bc = \frac{1}{288}b^2 \Rightarrow c = \frac{b}{17\times 24} = \frac{b}{408}$

$$\frac{17}{12}b - \frac{1}{408}b = \left(\frac{17\times 34 - 1}{408}\right)b = \frac{577}{408}b$$

$$\left(\frac{577}{408}\right)^2 = \left(\frac{17}{12} - \frac{1}{24\times 17}\right)^2 = 2 + \frac{1}{144} - \frac{1}{144} + \left(\frac{1}{24\times 17}\right)^2$$

for ~~This is a f~~ So $\left(\frac{577}{408}\right)^2$ is closer to 2.

We have $x_0 = 1$ $\quad x_1 = \frac{3}{2}$ $\quad x_2 = x_1 - \left(\frac{x_1^2-2}{2x_1}\right) = \frac{17}{12}$

$$x_3 = x_2 - \frac{(x_2^2-2)}{2x_2} = \frac{577}{408}$$

In general we can define $x_{n+1} = x_n - \frac{(x_n^2-2)}{2x_n}$ for all $n \geq 1$

$$\text{Then } x_{n+1}^2 = \left(x_n - \frac{(x_n^2-2)}{2x_n}\right)^2 = x_n^2 - 2x_n\left(\frac{x_n^2-2}{2x_n}\right) + \left(\frac{x_n^2-2}{2x_n}\right)^2$$

$$= 2 + \left(\frac{x_n^2-2}{2x_n}\right)^2$$

So $x_{n+1}^2 - 2 = \left(\frac{x_n^2-2}{2x_n}\right)^2$

We can find rational numbers $\frac{m}{n}$ with $\left(\frac{m}{n}\right)^2$ arbitrarily close to 2, but not exactly equal to 2.

What is a real number?

Legend has it that the Pythagorean brotherhood discovered that there is no rational number whose square is 2, and that this was so disturbing that the discoverer was thrown into the sea. It seems likely that most of those who did mathematics, across the world and across the ages, did not trouble themselves too much about the nature of number. In modern times, the nature of number was not a matter of concern until the 19th century. In fact Peano - who formulated the best-known axiomatisation of the natural numbers - died in 1936. When people did address the problem of number in the 19th century, what they came up with was strongly related to the Greek theory of proportion.

The two best known theories of real numbers are due to Dedekind and Cantor. We are going to look at Dedekind's theory.

Dedekind Cuts

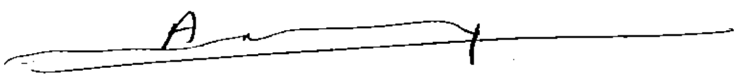
__Def^n__ A __Dedekind cut__ is a subset A of $\mathbb{Q}$ such that:

1. $A \neq \emptyset$
2. $A \neq \mathbb{Q}$
3. $a \in A \wedge a' \in \mathbb{Q} \wedge a' < a \Rightarrow a' \in A$
4. A does not have a maximal element, that is, $\nexists\, a \in A$ such that $a' \leq a \;\forall a' \in A$.

Dedekind did not make this last condition, but something like this is needed to make a proper identification between rational numbers and some of the Dedekind cuts.

Properties 2 and 3 imply that a Dedekind cut is __bounded above__ that is, if $A \subset \mathbb{Q}$ is a Dedekind cut then $\exists\, M \in \mathbb{Q}$ such that $a < M \;\forall a \in A$. To see this simply take any $M \in \mathbb{Q} \setminus A$ – which exists, by 2. Then property 3 implies $a < M \;\forall a \in A$.

A cut looks like this



But remember A is a set of __rational numbers__. Dedekind cuts are used to define the real numbers in terms of rational numbers.

72.1

Examples of sets with or without maximal elements

$A_1 = \{x \in \mathbb{R} : x \leq 1\}$ 1 is a maximal element

$A_2 = \{x \in \mathbb{Q} : x \leq 1\}$ Again, 1 is maximal

$A_3 = \{x \in \mathbb{Z} : x < 1\} = \{\ldots -3, -1, 0\}$ 0 is maximal

$A_4 = \{x \in \mathbb{Q} : x < 3\}$. There is no maximal element

To prove this explicitly, if $x < 3$ then

$x < \frac{x+3}{2} < 3 \left(= \frac{3+3}{2}\right)$ So $x \in A_4 \Rightarrow \frac{x+3}{2} \in A_4$

and x is not maximal. $x \in \mathbb{Q} \Rightarrow \frac{x+3}{2} \in \mathbb{Q}$

Similarly $\{x \in \mathbb{R} : x < 3\}$ does not have a maximal element

$A_5 = \{x \in \mathbb{Q} : x^2 + x \leq 2\}$

$x^2 + x \leq 2 \Leftrightarrow x^2 + x - 2 \leq 0 \Leftrightarrow (x+2)(x-1) \leq 0$

$\Leftrightarrow -2 \leq x \leq 1$ $1 \in A_5$ is maximal.

$A_6 = \{x \in \mathbb{Q} : x^2 + x \leq 5\}$

$x^2 + x \leq 5 \Leftrightarrow x^2 + x - 5 \leq 0$

$x^2 + x - 5 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{21}}{2}$ $\frac{-1 \pm \sqrt{21}}{2} \notin \mathbb{Q}$ 21 is not square of an integer

$x^2 + x \leq 5 \Leftrightarrow \frac{-1-\sqrt{21}}{2} \leq x \leq \frac{-1+\sqrt{21}}{2}$

A_6 does not have a maximal element.

Formal proof would use continuity of $x^2 + x - 5$

$A_7 = \{x \in \mathbb{R} : x^2 + x \leq 5\}$ does have a maximal element

because $\frac{-1+\sqrt{21}}{2} \in \mathbb{R}$ is maximal.

~~However $A_8 = \{x \in \mathbb{R} : x^2$~~

(72.2)

However $A_8 = \{x \in \mathbb{R} : x^2 + x < 5\}$ does not have a maximal element. If $x \in A_8$ then $\dfrac{x + \left(\frac{-1+\sqrt{21}}{2}\right)}{2} \in A_8$

and $x < \dfrac{x + \left(\frac{-1+\sqrt{21}}{2}\right)}{2}$

Which of the sets has a minimal element?

$a \in A$ is minimal if $a \le a'$ $\forall a' \in A$

A_5 and A_7 $\quad -2$ is minimal in A_5

$\frac{-1-\sqrt{21}}{2}$ is minimal in A_7