Examples of sets with or without maximal elements

$A_1 = \{x \in \mathbb{R} : x \leq 1\}$ 1 is a maximal element

$A_2 = \{x \in \mathbb{Q} : x \leq 1\}$ Again, 1 is maximal

$A_3 = \{x \in \mathbb{Z} : x < 1\} = \{\ldots -2, -1, 0\}$ 0 is maximal

$A_4 = \{x \in \mathbb{Q} : x < 3\}$. There is no maximal element

To prove this explicitly, if $x < 3$ then

$x < \frac{x+3}{2} < 3 \left(= \frac{3+3}{2}\right)$ So $x \in A_4 \Rightarrow \frac{x+3}{2} \in A_4$

and x is not maximal. $x \in \mathbb{Q} \Rightarrow \frac{x+3}{2} \in \mathbb{Q}$

Similarly $\{x \in \mathbb{R} : x < 3\}$ does not have a maximal element

$A_5 = \{x \in \mathbb{Q} : x^2 + x \leq 2\}$

$x^2 + x \leq 2 \Leftrightarrow x^2 + x - 2 \leq 0 \Leftrightarrow (x+2)(x-1) \leq 0$

$\Leftrightarrow -2 \leq x \leq 1$ $1 \in A_5$ is maximal.

$A_6 = \{x \in \mathbb{Q} : x^2 + x \leq 5\}$

$x^2 + x \leq 5 \Leftrightarrow x^2 + x - 5 \leq 0$

$x^2 + x - 5 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{21}}{2}$ $\frac{-1 \pm \sqrt{21}}{2} \notin \mathbb{Q}$ 21 is not squared an integer

$x^2 + x \leq 5 \Leftrightarrow \frac{-1-\sqrt{21}}{2} \leq x \leq \frac{-1+\sqrt{21}}{2}$

A_6 does not have a maximal element.

Formal proof could use continuity of $x^2 + x - 5$

$A_7 = \{x \in \mathbb{R} : x^2 + x \leq 5\}$ does have a maximal element

because $\frac{-1+\sqrt{21}}{2} \in \mathbb{R}$ is maximal.

~~However $A_8 = \{x \in \mathbb{R} : x^2$~~

However $A_8 = \{x \in \mathbb{R} : x^2 + x < 5\}$ does not have a maximal element. If $x \in A_8$ then $\frac{x + \left(\frac{-1+\sqrt{21}}{2}\right)}{2} \in A_8$

and $x < \frac{x + \left(\frac{-1+\sqrt{21}}{2}\right)}{2}$

Which of the sets has a minimal element?

$a \in A$ is minimal if $a \leq a'$ $\forall a' \in A$

A_5 and A_7 $\quad$ -2 is minimal in A_5

$\frac{-1-\sqrt{21}}{2}$ is minimal in A_7