PAPER CODE NO.
MATH 105(R)

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Numbers and Sets

TIME ALLOWED: Two and a half hours.

INSTRUCTIONS TO CANDIDATES: Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from equations and inequalities), and state whether each one is true.

a) $1 > 2 \wedge 3 < 6$.

b) $\neg\,(1 \leq 2 \vee 4 \leq 2)$.

c) $\forall x \in \mathbb{R}, x^3 > 0 \Rightarrow x > 0$.

d) $\forall x \in \mathbb{R},\ x^2 + x > 2$.

[7 marks]

2. Write down the negatives of the statements in 1, without using the negation symbol $\neg$.

[6 marks]

3. Let x_n be defined for integers $n \geq 0$ by $x_0 = 5$ and $x_{n+1} = 2x_n - 3$. Prove by induction that $x_n = 2^{n+1} + 3$ for all integers $n \geq 0$. [5 marks]

4. Find the prime factorisation of 1584. Hence or otherwise determine the *number* of divisors of 1584. You are *not* required to write them all down.

[5 marks]

5. Each of the following sets can be written as a single interval. Write down the interval in each case.

a) $(0, 2] \cap [1, 3] \cap [0, 4]$.

b) $((0, 2) \cap [1, 5)) \cup [2, 4]$.

c) $((0, 2] \cup [1, 3]) \setminus [2, 4]$.

[3 marks]

6. Let $m = 213$ and $n = 352$. Let d be the greatest common divisor of m and n. Using the Euclidean algorithm or otherwise:

(i) compute d;

(ii) find integers m_1 and n_1 such that $m = dm_1$ and $n = dn_1$;

(iii) find integers a and b such that $d = ma + nb$;

(iv) find the l.c.m. of m and n.

[9 marks]

7.

If both X and Y are sets of real numbers, state what it means for a function $f : X \to Y$ to be *strictly increasing* and what it means for $f : X \to Y$ to be *injective*. Prove that if $f : X \to Y$ is strictly increasing then it is injective. Determine whether $f : X \to Y$ is injective where:

a) $X = [0, \infty) = Y$ and $f(x) = x^2 + 1$;

b) $X = [0, \infty) = Y$ and $f(x) = (x - 1)^2$;

c) $X = (-\infty, 0) \cup (0, \infty)$ and $f(x) = \frac{1}{x}$.

[8 marks]

8. State the inclusion/exclusion formula for two finite sets A and B.

Suppose that in a class of 127 students, all of them are studying either Mathematics or Finance, but not necessarily both. Suppose that 105 of them are studying Mathematics and 56 are studying Finance. Determine how many students are studying both Mathematics and Finance.

[4 marks]

9. Define what it means for a real number x to be algebraic. Show that the following numbers are algebraic:

a) $2+\sqrt{2}$;

b) $\sqrt{2+\sqrt{2}}$.

[5 marks]

10. State without reasons which of the following sets is countable and which is uncountable.

a) The set of even integers.

b) The set of irrational real numbers.

c) The set of algebraic numbers.

[3 marks]

SECTION B

11.

(i) An equivalence relation $\sim$ on X (where $x \sim y$ means "x is equivalent to y") is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

Give the definition of the equivalence class $[x]$ of an element x of X.

(ii) Determine for which of the following $\sim$ is an equivalence relation on X.

a) $X = \mathbb{Z}$ and $m \sim n$ if and only if $m \geq n$.

b) $X = \mathbb{R}$ and $x \sim y$ if and only if $x - y \in \mathbb{Z}$.

(iii) Now let $X = \mathbb{Q} \setminus \{0\}$, define $x \sim y$ if and only if $x/y = 2^n$ for some $n \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation. Also, show that there are infinitely many distinct equivalence classes for $\sim$.

Hint: you may assume that there are infinitely many primes, and that any strictly positive *integer* can be written uniquely as a product of powers of distinct primes, up to order.

[15 marks]

12.

a) Prove by induction that $3n^2 + 5 < 2^n$ for all integers $n \geq 8$.

b) Prove by induction that for all $n \in \mathbb{Z}_+$, $n + 1 = 1 + n$, under the assumption that $(r + s) + 1 = r + (s + 1)$ for all r, $s \in \mathbb{Z}_+$.

Hence prove also by induction *on* m that $n + m = m + n$ for all $m \in \mathbb{Z}_+$, under the assumption that $(r + s) + t = r + (s + t)$ for all r, s, $t \in \mathbb{Z}_+$.

[15 marks]

13. Define what it means for a function $f : X \to Y$ to be *injective*, *surjective*, and a *bijection*. Prove that if $f : X \to Y$ and $g : Y \to Z$ are injective, then $g \circ f : X \to Z$ is also injective.

Define what it means for a set A to be *countable*.

Assuming that $\mathbb{Z}$ and $\mathbb{Z}^2$ are countable, prove by induction that $\mathbb{Z}^n$ is countable, for each $n \in \mathbb{Z}_+$.

Hint: $F : \mathbb{Z}^{n+1} \to \mathbb{Z}\times\mathbb{Z}^n$ is a bijection, where $F(m_1, \cdots m_{n+1}) = (m_1, (m_2, \cdots m_{n+1}))$.

[15 marks]

14. Define what it means for $A \subset \mathbb{Q}$ to be a *Dedekind cut*.

Determine which (if any) of the following is a Dedekind cut, checking whether or not the properties needed for a set to be a Dedekind cut do hold.

a) $A = \{x \in \mathbb{Q} : x^2 + x + 3 > 0\}$.

b) $A = \{x \in \mathbb{Q} : x^2 + x - 3 < 0\}$.

c) $A = \{x \in \mathbb{Q} : x^3 - x - 3 < 0\}$.

Hint: you may assume that if f is a polynomial then f is continuous, and, in particular, if $f(x) < 0$, then there is $n \in \mathbb{Z}_+$ such that $f(y) < 0$ whenever $|x - y| \leq 1/n$.

[15 marks]