PAPER CODE NO. **MATH 105(R)**

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SEPTEMBER 2013 EXAMINATIONS

Numbers and Sets

TIME ALLOWED: Two and a half hours.

INSTRUCTIONS TO CANDIDATES: Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from equations and inequalities), and state whether each one is true.

a) $-3 \in \mathbb{Z}$.

b) $\exists x \in \mathbb{R}, x^2 + x < -1$.

c) $\forall n \in \mathbb{Z}, \quad 2 \mid n \quad \Leftrightarrow \quad 2 \mid n^2$.

d) For all x, $y \in \mathbb{R}$

$$x \leq y \Leftrightarrow x^2 \leq y^2.$$

[7 marks]

2. Describe each of the following sets as a union of intervals

a) $\{x \in \mathbb{R} : x^2 \geq 9\}$.

b) $\left\{x \in \mathbb{R} : \dfrac{x}{x-2} > 2\right\}$.

[6 marks]

3. Prove by induction that $n^4 \leq 4^n$ for all integers $n \geq 4$. [6 marks]

4. Find all divisors of 1672.

[6 marks]

5. Prove that if m, n and p are any integers, and $m \mid n$ and $n \mid p$, then $m \mid p$.

[5 marks]

6. Let $m = 748$ and $n = 231$. Using the Euclidean algorithm or otherwise, find:

(i) the g.c.d. d of m and n;

(ii) integers m_1 and n_1 such that $m = dm_1$ and $n = dn_1$;

(iii) integers a and b such that $d = ma + nb$;

(iv) the l.c.m. of m and n.

[9 marks]

7. Write down what it means for a function $f : X \to Y$ to be injective. Define the image of a function $f : X \to Y$, and define what it means for $f : X \to Y$ to be a bijection. Find the image of f, and also determine whether f is injective, where $f : (0, \infty) \to \mathbb{R}$ is given by $f(x) = \dfrac{2x-1}{x+2}$.

[8 marks]

8. Determine which of the following numbers are rational. You may assume that $\sqrt{n}$ is not rational for any $n \in \mathbb{Z}_+$ which is not the square of another integer.

(i) $\dfrac{\sqrt{3}+2}{5}$.

(ii) The positive real number x such that $x = \dfrac{3x+5}{3x+1}$.

[5 marks]

9. State which of the following sets are countable:

(i) $[0, 1]$;

(ii) $\{n\pi : n \in \mathbb{Z}\}$;

(iii) $\mathbb{Q}$.

[3 marks]

SECTION B

10. a) An equivalence relation $\sim$ on X (where $x \sim y$ means "x is equivalent to y") is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

Give the definition of the equivalence class $[x]$ of an element x of X.

b) Determine for which of the following $\sim$ is an equivalence relation on X.

(i) $X = \mathbb{Z}$ and $m \sim n$ if and only if $m \leq n$.

(ii) $X = \mathbb{Z} \setminus \{0\}$ and $m \sim n$ if and only if $mn > 0$.

c) Now let $X = \mathbb{R}^2$. Show that $\sim$ is an equivalence relation on X, where $\sim$ is defined by

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 - x_2 + 2(y_1 - y_2) = 0.$$

Show also that each equivalence class of this equivalence relation is a straight line in $\mathbb{R}^2$. Write down the equivalence class of $(a, 0)$ for each $a \in \mathbb{R}$.

[15 marks]

11.

(i) Prove by induction on $n \in \mathbb{Z}_+$ that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(ii) Prove by induction on $n \in \mathbb{N}$ that for any natural number n, either n or $n-1$ is divisible by 2. Deduce that every natural number n is either of the form $2k$ for some $k \in \mathbb{N}$ or $2k+1$ for some $k \in \mathbb{N}$.

[15 marks]

12.

(i) State the inclusion/exclusion principle for two finite sets.

215 people visited the ice-cream shop. All of them had at least one scoop of ice-cream. 139 of them had a scoop of vanilla in their ice-cream. 164 had a scoop of another flavour (and may also have had vanilla).

(ii) How many had a scoop of vanilla and also a scoop of another flavour?

(iii) How many had only vanilla?

(iv) How many did not have vanilla?

Of these 215 people, 28 had scoops of both vanilla and chocolate, and 31 had neither.

(v) How many had a scoop of chocolate?

(vi) How many had a scoop of chocolate and not vanilla?

(vii) Deduce from the inclusion/exclusion principle for two sets, that if V, C and O denote the sets of people who had vanilla, chocolate or another flavour, then

$$|V \cap (C \cup O)| = |V \cap C| + |V \cap O| - |V \cap C \cap O|.$$

Of these 215 people, 75 had vanilla and something other than vanilla or chocolate (and may also have had chocolate).

(viii) How many people had vanilla, chocolate, and something else?

[15 marks]

13. Define what it means for $A \subset \mathbb{Q}$ to be a *Dedekind cut.*

Determine which (if any) of the following is a Dedekind cut, checking whether or not the properties needed for a set to be a Dedekind cut do hold.

Hint You may find it helpful to show that, for any number c, the polynomial $f(x) = x^3 - 3x + c$ is strictly increasing for $x \leq -2$. You may use that this polynomial f is continuous, which implies that, if $f(x) < 0$, there is a $\delta > 0$ (which we can take to be rational) such that $f(y) < 0$ for all y with $x - \delta \leq y \leq x + \delta$.

(i) $A = \{x \in \mathbb{Q} : 5x + 1 > 0\}$.

(ii) $A = \{x \in \mathbb{Q} : x^3 - 12x + 1 < 0\}$.

(iii) $A = \{x \in \mathbb{Q} : x^3 - 12x + 20 < 0\}$.

(iv) $A = \{x \in \mathbb{Q} : x^3 - 12x + 1 < 0\} \cap \{x \in \mathbb{Q} : x < -2\}$.

[15 marks]