UNIVERSITY OF LIVERPOOL

MATH 105(R)

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Two and a half hours.

Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from the equation), and determine whether each one is true.

a) For $x \in \mathbb{R}$, $(x^2 + 2x - 3 = 0) \Leftrightarrow (x = 1 \vee x = -3)$

b) For $x \in \mathbb{R}$, $(x > 2) \Rightarrow (x > 1 \wedge x < 3)$

[6 marks]

2. Negate each of the following statements, using logical symbols where possible.

a) $\forall x \in \mathbb{R},\ x \in \mathbb{Q}$.

b) $\forall x \in \mathbb{R},\ x < 3 \Rightarrow x^2 < 9$.

[4 marks]

3. In each of the following, state whether the element x is in the set X

a) $x = 2$, $X = [0, 2)$.

b) $x = 3$, $X = [0, \sqrt{5}]$.

c) $x = 1$, $X = (1, 5)$.

d) $x = 3$, $X = (-\infty, 3]$.

e) $x = 1 - 2i$, $X = \mathbb{R}$.

f) $x = \sqrt{3}$, $X = \mathbb{Q}$.

[6 marks]

4. In each of the following, find the set of all $x \in \mathbb{R}$ satisfying the inequalities.

a) $1 < 3x - 5$

b) $-1 < \dfrac{2 - x}{1 + x} < 1$, $x \neq -1$.

[5 marks]

5. Show by induction that $2^n < n!$ for all integers $n \geq 4$.

[6 marks]

6. Let $a = 168$ and $b = 408$. Using the Euclidean algorithm or otherwise, find:

(i) the g.c.d. d of a and b;

(ii) integers r and s such that $a = dr$ and $b = ds$;

(iii) integers m and n such that $d = ma + nb$;

(iv) the l.c.m. of a and b.

[9 marks]

7. Determine the images of the following functions, and also determine whether the functions are injective, surjective or neither.

a) $f : (-1, \infty) \to (-1, \infty)$ given by $f(x) = x^3$.

b) $f : (-\infty, 1) \cup (1, \infty) \to \mathbb{R}$ given by $f(x) = \dfrac{x+1}{x-1}$.

[7 marks]

8.

a) Find a conditional definition of the set $\{\sin x : x \in \mathbb{R}\}$.

b) Find a constructive definition for the set

$$\{x \in \mathbb{R} : -2 \leq x \leq 2\}.$$

[6 marks]

9. Determine which of the following sequences are increasing, which are decreasing, and which are neither.

a) $x_n = (-1)^n n$, $n \geq 1$.

b) $x_n = n^2 - 8n + 15$, $n \geq 1$.

c) $x_n = 2 - \dfrac{1}{n^2}$, $n \geq 1$.

[6 marks]

SECTION B

10. An equivalence relation $\sim$ on X – where $x \sim y$ means "x is equivalent to y" is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

In each of the following cases, determine whether $\sim$ is an equivalence relation on X.

a) $X = \mathbb{Z}$ and $x \sim y \Leftrightarrow 3|x - y$.

b) $X = \mathbb{Z}$ and $x \sim y \Leftrightarrow xy$ is even.

c) $X = \mathbb{Q}$ and $x \sim y \Leftrightarrow xy \in \mathbb{Z}$

d) $X = \mathbb{C}$ and $x \sim y \Leftrightarrow x - y = m + ni$ for m, $n \in \mathbb{Z}$

[15 marks]

11. Define x_n inductively for $n \in \mathbb{N}$ by

$$x_0 = 2, \quad x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{x_n}{2} + \frac{3}{2x_n}$$

Prove the following.

(i) $x_n > 0$ for $n \geq 0$. (*Hint*: use induction.)

(ii) $x_{n+1}^2 - 3 = \dfrac{(x_n^2 - 3)^2}{4x_n^2}$ for $n \geq 0$. (*Hint*: use the formula for x_{n+1}.)

(iii) $x_n^2 - 3 > 0$ for $n \geq 0$. (*Hint*: use induction and (ii).)

(iv) $1 \leq x_{n+1} < x_n$ for $n \geq 0$. (*Hint*: use the formula for x_{n+1} and use (iii) for all $n \in \mathbb{N}$.)

[15 marks]

12. An outdoor activity centre has organised a weekend with three different activities available: abseiling, canoeing and swimming. 30 people book for the weekend, and all of them do at least one of the activities on offer. 24 of them do abseiling, 26 do canoeing and 26 sail. All of those who do abseiling do at least one other activity and all but one of those who sail does at least one other activity. 17 people do all three activities.

Let A, C and S be the sets of people who abseil, canoe and sail respectively.

(i) State the inclusion-exclusion principle for three sets. You may call these sets A, C and S, and it might be convenient to do so.

(ii) Remembering that everyone who abseils also does one of the other two activities, write $|A|$ in terms of $|A \cap C|$, $|A \cap S|$ and $|A \cap C \cap S|$.

Hint Use the inclusion-exclusion principle for two sets.

iii) Find the number of people who both canoe and sail, that is, $|C \cap S|$.

Hint: Use the formulae from (i) and (ii).

(iv) Find $|A \cap S|$ and $|A \cap C|$.

Hint Remember that all but one of the people who sail also do at least one other activity, and use this to write $|S|$ in terms of $|A \cap S|$, $|C \cap S|$ and $|A \cap C \cap S|$.

[15 marks]

13.

(i) Give the definition of a *Dedekind cut*

(ii) Determine which (if any) of the following sets A are Dedekind cuts. Give brief reasons for your answers.

a) $A = \{x \in \mathbb{Q} : x > 1\}$

b) $A = \{x \in \mathbb{Q} : x \leq 2\}$

(iii) Show that $A = \{x \in \mathbb{Q} : x^2 + x - 1 < 0 \vee x < 0\}$ is bounded above by 1 and has no maximal element, and hence, or otherwise, show that A is a Dedekind cut.

Hint Show that $x^2 + x - 1$ is strictly increasing on $[-\frac{1}{2}, \infty)$, and show that if $a \in A$ with $0 < a < 1$, and $0 < \varepsilon < 1$ then $(a + \varepsilon)^2 + a + \varepsilon - 1 < a^2 + a - 1 + 4\varepsilon$. Hence show that if it is also true that $\varepsilon \in \mathbb{Q}$ and $\varepsilon < -\dfrac{a^2 + a - 1}{4}$ then $a + \varepsilon \in A$.

[15 marks]