PAPER CODE NO.
MATH 105

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JANUARY 2013 EXAMINATIONS

Numbers and Sets

TIME ALLOWED: Two and a half hours.

INSTRUCTIONS TO CANDIDATES: Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from equations and inequalities), and state whether each one is true.

a) $1.5 \in \mathbb{Z}$.

b) $(x \in \mathbb{R} \wedge x^2 \leq 0) \Rightarrow x = 0$.

c) $\forall n \in \mathbb{Z}, \quad 2 \mid n \ \bigvee \ 2 \mid (n-1)$.

d) For any real number x, $0 \leq x \leq 1 \Leftrightarrow 0 \leq x^2 \leq 1$.

[7 marks]

2. Describe each of the following sets as a union of intervals:

a) $\{x \in \mathbb{R} : x^2 < 9\}$;

c) $\left\{x \in \mathbb{R} : \dfrac{x}{x+2} > 3\right\}$.

[6 marks]

3. Prove by induction that $n^3 \leq 3^n$ for all integers $n \geq 3$. [6 marks]

4. Find all divisors of 1989.

[6 marks]

5. Prove that is n is any integer, then n is even if and only if n^2 is even. You may assume that an integer n is not even if and only if $n-1$ is even.

[5 marks]

6. Let $m = 623$ and $n = 231$. Using the Euclidean algorithm or otherwise, find:

(i) the g.c.d. d of m and n;

(ii) integers m_1 and n_1 such that $m = dm_1$ and $n = dn_1$;

(iii) integers a and b such that $d = ma + nb$;

(iv) the l.c.m. of m and n.

[9 marks]

7. Write down what it means for a function $f : X \to Y$ to be injective. Define the image of a function $f : X \to Y$, and define what it means for $f : X \to Y$ to be a bijection. Find the image of f, and determine whether f is injective, where $f : (0, \infty) \to (0, \infty)$ is given by $f(x) = \dfrac{x+1}{x+2}$.

[8 marks]

8. Determine which of the following numbers are rational. You may assume that $\sqrt{n}$ is not rational for any $n \in \mathbb{Z}_+$ which is not the square of another integer.

(i) The real number x such that $\dfrac{x+2}{x+3} = 10$.

(ii) The positive real number y such that

$$\frac{2}{y+2} = 1 - \frac{1}{y+1}.$$

[5 marks]

9. State which of the following sets are countable:

(i) $\{n \in \mathbb{Z} : n \text{ is even }\}$;

(ii) $(0, \infty)$;

(iii) $\{0\}$.

[3 marks]

SECTION B

10. a) An equivalence relation $\sim$ on X (where $x \sim y$ means "x is equivalent to y") is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

Give the definition of the equivalence class $[x]$ of an element x of X.

b) Determine for which of the following $\sim$ is an equivalence relation on X.

(i) $X = \mathbb{Z}$ and $m \sim n$ if and only if $m = n + 3k$ for some $k \in \mathbb{Z}$.

(ii) $X = \mathbb{Z}$ and $m \sim n$ if and only if $m = n + 3k + 1$ for some $k \in \mathbb{Z}$.

c) Now let X be the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real coefficients a, b, c, d. Show that $\sim$ is an equivalence relation on X, where $\sim$ is defined by:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \quad \Leftrightarrow \quad c_1 = c_2.$$

Write down the equivalence class of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

[15 marks]

11.

(i) Prove by induction on $n \in \mathbb{Z}_+$ that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

(ii) Prove by induction on $n \in \mathbb{N}$ that for any natural number n, exactly one of n, $n-1$ or $n+1$ is divisible by 3. Hence, or otherwise, show that any natural number n is of the form $3k + r$, for some $k = \mathbb{N}$ and $r = 0$ or ± 1.

[15 marks]

12.

(i) State the inclusion/exclusion principle for two finite sets.

150 people visited a supermarket. Everyone bought at least one item. 66 of them bought bread. 128 of them bought another item (and may also have bought bread).

(ii) How many bought bread and something else?

(iii) How many only bought bread?

(iv) How many did not buy bread?

Of these 150 people, 31 bought both milk and bread, and 56 bought neither.

(v) How many bought milk?

(vi) How many bought milk and not bread?

(vii) Deduce from the inclusion/exclusion principle for two sets, that if B, M and O denote the sets of people who bought bread, milk, or another item, then

$$|B \cap (M \cup O)| = |B \cap M| + |B \cap O| - |B \cap M \cap O|.$$

Of these 150 people, 42 bought bread and something other than bread or milk (and may also have bought milk).

(viii) How many people bought bread, milk, and something else?

[15 marks]

13. Define what it means for $A \subset \mathbb{Q}$ to be a *Dedekind cut.*

Determine which (if any) of the following is a Dedekind cut, checking whether or not the properties needed for a set to be a Dedekind cut do hold.

Hint You may find it helpful to show that, for any number c, the polynomial $f(x) = x^3 - 3x + c$ is strictly increasing for $x \leq -1$. You may use that this polynomial f is continuous, which implies that, if $f(x) < 0$, there is a $\delta > 0$ (which we can take to be rational) such that $f(y) < 0$ for all y with $x - \delta \leq y \leq x + \delta$.

(i) $A = \{x \in \mathbb{Q} : 3x \leq 2\}$.

(ii) $A = \{x \in \mathbb{Q} : x^3 - 3x + 1 < 0\}$.

(iii) $A = \{x \in \mathbb{Q} : x^3 - 3x + 3 < 0\}$.

(iv) $A = \{x \in \mathbb{Q} : x^3 - 3x + 1 < 0\} \cap \{x \in \mathbb{Q} : x < -1\}$.

[15 marks]