UNIVERSITY OF LIVERPOOL

MATH 105

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Two and a half hours.

Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from equations and and inequalities), and determine whether each one is true.

a) $x \in \mathbb{R} \Rightarrow x^2 + 2x + 1 \geq 0$.

b) $\exists n \in \mathbb{Z}$ such that $n \leq p \ \forall p \in \mathbb{Z}$. [6 marks]

2. Negate each of the following statements, using logical symbols where possible.

a) $x < 0 \vee x \geq 2$.

b) $\sin x < x \ \forall x \in (0, 1)$.

[4 marks]

3. Identify which of the following sets is equal to the empty set. Write each of the others as either a single interval — if this is possible — or a union of two intervals.

a) $[-3, 7] \cap (-1, 8) \cap [-4, 6]$

b) $([-4, 1] \cup (3, 8]) \cap [2, 3]$.

c) $([-5, 1] \cup (3, 7)) \cap ([-3, 0] \cup (2, 6))$.

[6 marks]

4. In each of the following, find the set of all $x \in \mathbb{R}$ satisfying the inequalities. The set might be empty.

a) $x^2 + x > 2$.

b) $x^2 + x + 2 < 0$.

[6 marks]

5. Let x_n be defined inductively for $n \in \mathbb{N}$ by $x_0 = 1$ and $x_{n+1} = 3x_n - 4$. Prove by induction that $x_n = 2 - 3^n$ for all $n \in \mathbb{N}$.

[5 marks]

6. Let $a = 572$ and $b = 385$. Using the Euclidean algorithm or otherwise, find:

(i) the g.c.d. d of a and b;

(ii) integers r and s such that $a = dr$ and $b = ds$;

(iii) integers m and n such that $d = ma + nb$;

(iv) the l.c.m. of a and b.

[9 marks]

7. Write down what it means for a function $f : X \to Y$ to be injective. Define the image of a function $f : X \to Y$, and define what it means for $f : X \to Y$ to be a bijection. Find the images of the following functions, and also determine whether the functions are injective:

a) $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x^{-2}$;

b) $f : [-\pi/2, \pi/2] \to \mathbb{R}$ given by $f(x) = \sin^2 x$.

[10 marks]

8. State the inclusion/exclusion principle for two sets A_1 and A_2.

10 web retailers are selling Series 1 or Series 2 Dr Who DVD's. Series 1 is sold by 9 of them and Series 2 by 8.

(i) Determine how many of the retailers are selling both Series 1 and Series 2, and how many sell just Series 1, or just Series 2.

(ii) Now suppose that 7 of these 10 retailers also sell Series 3, and that 6 retailers sell all three series. Determine how many sell exactly two of the three series.

Hint: Only the inclusion/exclusion principle for A_1 and A_2 is needed.

[9 marks]

SECTION B

9. An equivalence relation $\sim$ on X (where $x \sim y$ means "x is equivalent to y") is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

(i) Let $X = \mathbb{Z}$ and let $\sim$ be the relation defined on X by $m \sim n \Leftrightarrow m - n$ is even. Show that $\sim$ is an equivalence relation on X, determine the number of equivalence classes, and write down a representative of each equivalence class.

(ii) Let X be the set of all polynomials $f(x)$ of degree at most one with integer coefficients, that is

$$X = \{a_0 + a_1 x : a_0, a_1 \in \mathbb{Z}\}.$$

Define $f(x) \sim g(x) \Leftrightarrow f(x) - g(x) = c_0 + c_1 x$ for even integers c_0 and c_1. Show that $\sim$ is an equivalence relation. Determine the number of equivalence classes of $\sim$, and write down a representative of each equivalence class.

[15 marks]

10. Let x_n be defined inductively for $n \in \mathbb{N}$ by

$$x_0 = 1, \quad x_{n+1} = 3 - \frac{7}{3 + x_n}.$$

(i) Prove by induction that $1 \leq x_n < 2$ for all $n \in \mathbb{N}$.

(ii) Show that

$$x_{n+2} - x_{n+1} = \frac{7(x_{n+1} - x_n)}{(3 + x_n)(3 + x_{n+1})},$$

and hence, or otherwise, show by induction that x_n is an increasing sequence.

(iii) Prove by induction that, for all $n \in \mathbb{N}$,

$$|x_{n+1} - x_n| \leq \left(\frac{7}{16}\right)^n \frac{1}{4}.$$

[15 marks]

11. Define what it means for $A \subset \mathbb{Q}$ to be a *Dedekind cut.*

Determine which (if any) of the following is a Dedekind cut.

a) $A = \{x \in \mathbb{Q} : x \leq 6.5\}$;

b) $A = \{x \in \mathbb{Q} : 7 < x\}$;

c) $A = \{x \in \mathbb{Q} : x^2 - 3x + 1 < 0\}$.

Now let A be a Dedekind cut, and define

$$2A = \{2x : x \in A\}, \quad -A = \{-x : x \in A\}.$$

Show that $2A$ is a Dedekind cut and that $-A$ is not one.

[15 marks]

12. Define what it means for a set A to be *finite*, *of cardinality* n, and what it means for A to be *countable.*

State which of the following sets A, B, C and D are countable and which are uncountable. No proofs are required.

a) $A = [0, 1]$.

b) $B = [0, 1)$.

c) $C = \mathbb{Z}$.

d) $D = \mathbb{N}^2$.

Show that there is an injective map $g : A \to B$ and an injective map $h : B \to A$. State the Schröder-Bernstein Theorem, and explain how it applies for A and B as above.

By considering $\{(m, n) \in \mathbb{N}^2 : m+n = p\}$ or otherwise, write $\mathbb{N}^2$ as a countable union of finite sets.

[15 marks]