UNIVERSITY OF LIVERPOOL

MATH 105

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Two and a half hours.

Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from the equation), and determine whether each one is true.

a) For $x \in \mathbb{R}$, $(x^2 + x - 2 = 0) \Leftrightarrow (x = 1 \vee x = -2)$.

b) For $x \in \mathbb{R}$, $(x > 0 \wedge x < 3) \Rightarrow (x > 1 \wedge x < 2)$.

[6 marks]

2. Negate each of the following statements, using logical symbols where possible.

a) $\forall x \in \mathbb{R},\ x^2 + x + 1 > 0$.

b) $\forall x,\ y \in \mathbb{R},\ x > y \Rightarrow x^2 > y^2$.

[4 marks]

3. In each of the following, state whether the element x is in the set X.

a) $x = 3$, $X = [0, \pi]$.

b) $x = 3$, $X = [0, 4] \cap [1, 2]$.

c) $x = 3$, $X = (-\infty, 3)$.

d) $x = 2$, $X = [-2, 2]$.

e) $x = 2 + 3i$, $X = \mathbb{R}$.

f) $x = \pi$, $X = \mathbb{Z}$. [6 marks]

4. In each of the following, find the set of all $x \in \mathbb{R}$ satisfying the inequalities.

a) $1 - 3x > 5$.

b) $2 < \dfrac{1 + x}{1 - x} < 3$, $x \neq 1$.

[5 marks]

5. Show by induction that $n^3 < 2^n$ for all integers $n \geq 10$.

[6 marks]

6. Let $a = 351$ and $b = 279$. Using the Euclidean algorithm or otherwise, find:

(i) the g.c.d. d of a and b;

(ii) integers r and s such that $a = dr$ and $b = ds$;

(iii) integers m and n such that $d = ma + nb$;

(iv) the l.c.m. of a and b.

[9 marks]

7. Determine the images of the following functions, and also determine whether the functions are injective, surjective or neither.

a) $f : (-1, \infty) \to (-1, \infty)$ given by $f(x) = x^2$.

b) $f : (-\infty, -1) \cup (-1, \infty) \to \mathbb{R}$ given by $f(x) = \dfrac{x}{x+1}$.

[7 marks]

8.

a) Find a conditional definition of the set $\{x^2 + 1 : x \in \mathbb{R}\}$.

b) Find a constructive definition for the set

$$\{n \in \mathbb{Z}_+ : n \geq 2 \text{ and 3 is the only prime factor of n}\}.$$

[6 marks]

9. Determine which of the following sequences are increasing, which are decreasing, and which are neither.

a) $x_n = n^3$, $n \geq 1$.

b) $x_n = n^2 - 7n + 10$, $n \geq 1$.

c) $x_n = \dfrac{n^2 - 1}{n^2 + 1}$, $n \geq 1$.

[6 marks]

SECTION B

10. An equivalence relation $\sim$ on X (where $x \sim y$ means "x is equivalent to y") is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

In each of the following cases, determine whether $\sim$ is an equivalence relation on X.

a) $X = \mathbb{N}$ and $x \sim y \Leftrightarrow x - y \in \mathbb{N}$.

b) $X = \mathbb{N}$ and $x \sim y \Leftrightarrow x + y$ is even.

c) $X = \mathbb{N}$ and $x \sim y \Leftrightarrow x + y$ is divisible by 3.

d) X is the set of 2×2 matrices with real coefficients, and
$A \sim B \Leftrightarrow$ there is an invertible 2×2 matrix P such that $B = PAP^{-1}$.

[15 marks]

11.

a) Show by induction on n, or otherwise, that for any real number $x \neq 1$ and any integer $n \geq 0$,

$$1 + \cdots + x^n = \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}.$$

b) Show by induction on n, or otherwise, that for any real number $x \neq 1$ and any integer $n \geq 1$,

$$x + \cdots + nx^n = \sum_{k=1}^{n} kx^k = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}.$$

[15 marks]

12. In a restaurant, 37 people are eating dinner. Three courses are available: starter, main course and dessert. Everyone has at least one course, and 33 people have at least two courses. 27 people have the starter, 36 people have the main course, and 21 have dessert. All of these might also have other courses. Also, 4 people only have the main course and 26 people have both the starter and the main course. Some of these people may have dessert as well.

Let S, M and D be the sets of people having starter, main course and dessert respectively.

(i) State the inclusion-exclusion principle for three sets. You may call these sets S, M and D.

(ii) Show that the number of people having at least two courses is

$$|S \cap M| + |M \cap D| + |D \cap S| - 2|M \cap S \cap D|$$

Hint: Apply the inclusion-exclusion principle to the three sets $S \cap M$, $M \cap D$ and $S \cap D$.

iii) Find the number of people having all three courses.

Hint: Add equations from (i) and (ii)

(iv) Find the number of people having both the main course and the dessert.

Hint: Apply the inclusion exclusion principle to the sets $M \cap S$, $M \cap D$.

[15 marks]

13.

(i) Give the definition of a *Dedekind cut*

(ii) Determine which (if any) of the following sets A are Dedekind cuts. Give brief reasons for your answers.

a) $A = \mathbb{Q}$

b) $A = \{x \in \mathbb{Q} : x^2 < 1\}$

(iii) Show that $A = \{x \in \mathbb{Q} : x^3 < 2\}$ has no maximal element, and hence, or otherwise, show that A is a Dedekind cut.

Hint: If $1 \leq a$, and $0 < \varepsilon < 1$ show that $(a + \varepsilon)^3 < a^3 + 7a^2\varepsilon$ and hence, if it is also true that $a \in A$, and $\varepsilon \in \mathbb{Q}$ and $\varepsilon < \dfrac{2 - a^3}{7a^2}$ then $a + \varepsilon \in A$. [15 marks]