MATH105 Sheet 8: Solutions to Practice Problems

5.

a) (i) $x - x = 0 \in \mathbb{Z}$, so $x \sim x$ for all $x \in \mathbb{R}$ and $\sim$ is *reflexive.*

(ii) $x - y \in \mathbb{Z} \Leftrightarrow y - x \in \mathbb{Z}$). So, for all x, $y \in \mathbb{R}$, $x \sim y \Leftrightarrow y \sim x$, and $\sim$ is *symmetric*

(iii)
$$x - y \in \mathbb{Z} \wedge y - z \in \mathbb{Z} \Rightarrow x - z = (x - y) + (y - z) = x - z \in \mathbb{Z}.$$

So, for all x, y, $z \in \mathbb{Z}$,
$$x \sim y \wedge y \sim z \Rightarrow x \sim z,$$

and $\sim$ is *transitive.*

Hence $\sim$ is an equivalence relation.

b) If $x = 1$ and $y = 0$ then $x - y = 1 \in \mathbb{N}$ and hence $1 \sim 0$. But $0 - 1 = -1 \notin \mathbb{N}$ and hence it is not true that $0 \sim 1$. So $\sim$ is not symmetric, and is not an equivalence relation on $\mathbb{R}$.

c) If $m = 2$ and $n = 1$ then $m/n = 2 \in \mathbb{Z}_+$ and so $2 \sim 1$. But $n/m = 1/2 \notin \mathbb{Z}_+$ and so it is not true that $1 \sim 2$. So $\sim$ is not symmetric, and is not an equivalence relation on $\mathbb{Z}_+$.

6.

(i) For any m, $n \in \mathbb{Z}$, $m - m = 0$ and $n - n = 0$, and 0 is even. So $(m, n) \sim (m, n)$ and $\sim$ is *reflexive.*

(ii) For any m_1, n_1, m_2, $n_2 \in \mathbb{Z}$,

$$(m_1, n_1) \sim (m_2, n_2) \quad \Rightarrow \quad (m_1 - m_2 \text{ even } \wedge n_1 - n_2 \text{ even }) \quad \Rightarrow \quad (m_2 - m_1 \text{ even } \wedge n_2 - n_1 \text{ even })$$
$$\Rightarrow \quad (m_2, n_2) \sim (m_1, n_1).$$

So $\sim$ is *symmetric*

(iii) For any m_1, n_1, m_2, n_2, m_3, $n_3 \in \mathbb{Z}$,

$$((m_1, n_1) \sim (m_2, n_2) \wedge (m_2, n_2) \sim (m_3, n_3)) \quad \Rightarrow \quad (m_1 - m_2 \text{ even } \wedge n_1 - n_2 \text{ even } \wedge m_2 - m_3 \text{ even } \wedge n_2 - n_3 \text{even})$$
$$\Rightarrow \quad (m_2 - m_1 + m_3 - m_2 \text{ even } \wedge n_2 - n_1 + n_3 - n_2 \text{ even })$$
$$\Rightarrow \quad (m_3 - m_1 \text{ even } \wedge n_3 - n_1 \text{ even }) \quad \Rightarrow \quad (m_1, n_1) \sim (m_3, n_3).$$

So $\sim$ is *transitive.*

So $\sim$ is an equivalence relation. If we consider the four vectors

$$(0, 0), \quad (1, 0), \quad (0, 1), \quad (1, 1),$$

then these are in four different equivalence classes. Denoting the equivalence class of a vector (m, n) by $[(m, n)]$, we have

$$[(0, 0)] = \{(2k, 2\ell) : k, \ell \in \mathbb{Z}\},$$
$$[(1, 0)] = \{(2k + 1, 2\ell) : k, \ell \in \mathbb{Z}\},$$
$$[(0, 1)] = \{(2k, 2\ell + 1) : k, \ell \in \mathbb{Z}\},$$
$$[(1, 1)] = \{(2k + 1, 2\ell + 1) : k, \ell \in \mathbb{Z}\}.$$

7.

a) We have $15 = 3 \times 5$ and $35 = 5 \times 7$ and 11 is prime. So the l.c.m. n of 11, 15 and 35 is $11 \times 3 \times 5 \times 7 = 11 \times 105 = 1155$.

b) The l.c.m. of 15 and 35 is $15 \times 7 = 3 \times 35 = 105$. So we have

$$\frac{b_1}{15} + \frac{c_1}{35} = \frac{7b_1 + 3c_1}{105}.$$

One solution to $7b_1 + 3c_1 = 1$ is $b_1 = 1$ and $c_1 = -2$. Then

$$\frac{a}{11} + \frac{e}{105} = \frac{105a + 11e}{1155} = \frac{1}{1155} \Leftrightarrow 105 + 11e = 1.$$

To solve this using the Euclidean algorithm,

$$\begin{array}{cc|c}1 & 0 & 105\\ 0 & 1 & 11\end{array} \quad \begin{array}{c} R_1 - 9R_2 \\ \rightarrow \end{array} \quad \begin{array}{cc|c}1 & -9 & 6\\ 0 & 1 & 11\end{array} \quad \begin{array}{c} \rightarrow \\ R_2 - R_1 \end{array} \quad \begin{array}{cc|c}1 & -9 & 6\\ -1 & 10 & 5\end{array} \quad \begin{array}{c} R_1 - R_2 \\ \rightarrow \end{array} \quad \begin{array}{cc|c}2 & -19 & 1\\ -1 & 10 & 5\end{array}$$

So we can take $a = 2$ and $e = -19$ which gives

$$a = 2, \quad b = -19, \quad c = 38.$$

Thus

$$\frac{2}{11} - \frac{19}{15} + \frac{38}{35} = \frac{1}{1155}.$$

Of course there are many other solutions.

8.

a) Suppose for contradiction that there are $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$ such that

$$\frac{1}{7} + \frac{5}{4}\sqrt{3} = \frac{p}{q}$$

Then

$$\sqrt{3} = \frac{4p}{5q} - \frac{4}{35} = \frac{28p - 20q}{35q} \in \mathbb{Q}$$

which is a contradiction.

b) Suppose for contradiction that $x \in \mathbb{Q}$ with $a + b\sqrt{3} = x$. Since $b \neq 0$, $b^{-1} \in \mathbb{Q}$ exists and $\sqrt{3} = (x - a) \cdot b^{-1} \in \mathbb{Q}$ which is again a contradiction.

c) If $3^{1/6} = x \in \mathbb{Q}$ then $\sqrt{3} = x^3 \in \mathbb{Q}$, which is a contradiction.

9. $3^0 + (-1)^0 = 1 + 1 = 2 = 3^1 + (-1)^1$, the formula $x_k = 3^k + (-1)^k$ holds for $k = 0$ and $k = 1$.
Now suppose the formula holds for $k \leq n$. Then

$$x_{n+1} = 2x_n + 3x_{n-1} = 2(3^n + (-1)^n) + 3(3^{n-1} + (-1)^{n-1}) = 2 \cdot 3^n + 3^n + 2 \cdot (-1)^n - 3 \cdot (-1)^n$$

$$= 3^{n+1} + (-1)^{n+1}$$

So if the formula holds for $k \leq n$ then it also holds for $k = n + 1$, and hence for $k = n + 1$.
So by induction $x_n = 3^n + (-1)^n$ for all n.

As noted in the question, we need two *bases cases, and the most natural ones to take are $n = 0$ and $n = 1$. Also, as noted in the question, we need to assume* at least the two cases $k = n - 1$ *and $k = n$ in order to get the case $k = n + 1$. In the solution above I have chosen to make the inductive hypothesis "Assume true for $k \leq n$", to deduce the case for $k = n + 1$, which means that "True for $k \leq n$ implies "True for $k \leq n + 1$".*