MATH105 Solutions to Practice Problems 6

7.

a) $[1,3) \cap (2,4] = \{x \in \mathbb{R} : 1 \le x < 3 \wedge 2 < x \le 4\} = (2,3)$.

b) $[1,3) \cup (2,4] = \{x \in \mathbb{R} : 1 \le x < 3 \vee 2 < x \le 4\} = [1,4]$.

c) $([2,5] \cup [1,4]) \setminus (0,3) = \{x \in \mathbb{R} : (2 \le x \le 5 \vee 1 \le x \le 4) \wedge \neg\, (0 < x < 3) = \{x \in \mathbb{R} : (1 \le x \le 5) \wedge (x \le 0 \vee x \ge 3\} = [3,5]$.

d) $[2,5] \cup ([1,4] \setminus (3,4) = \{x \in \mathbb{R} : (2 \le x \le 5 \vee 1 \le x \le 4) \wedge \neg\, (3 < x < 4) = \{x \in \mathbb{R} : (1 \le x \le 5) \wedge (x \le 3 \vee x \ge 4\} = [1,3] \cup [4,5]$.

8.

a) $x^4 = (x^2)^2 \ge 0$ for all $x \in \mathbb{R}$, so the image of f is contained in $[0,\infty)$. Conversely, if $y \ge 0$ then $y^{1/4} = \sqrt{\sqrt{y}}$ exists and $(y^{1/4})^4 = y$. So the image of f is $[0,\infty)$.

b) $f(x) = y \Leftrightarrow x - 2 = y \Leftrightarrow x = y + 2$. So $\mathrm{Im}(f) = \mathbb{R}$.

c) Since $e^x > 0$ for all $x \in \mathbb{R}$, the image of f is contained in $(0,\infty)$. For any $y > 0$,

$$e^x = y \Leftrightarrow x = \ln y.$$

So $\mathrm{Im}(f) = (0,\infty)$.

d)

$$y = 1 + \frac{1}{x} \Leftrightarrow x = \frac{1}{y-1}$$

Since $\dfrac{1}{y-1}$ is defined for $y \ne 1$, the image of f is $(0,1) \cup (1,\infty) = \{y \in (0,\infty : y \ne 1\}$.

9.

a) This function f is not injective because, for example, $1^4 = (-1)^4$. It is not surjective because the image $[0,\infty)$ is not equal to the codomain $\mathbb{R}$.

b) This function f is injective and surjective because $f(x) = y \Leftrightarrow x = y + 2$. So for each y there is at most one x with $f(x) = y$ (injective) and at least one x with $f(x) = y$ (surjective). Therefore it is also a bijection, since it is both injective and surjective.

c) Since $f(x) = y \Leftrightarrow x = \ln y$, this function f is injective. Since the image $(0,\infty)$ is not equal to the codomain $\mathbb{R}$, the function is not surjective. Therefore, it is not a bijection.

d)

e) Since $f(x) = y \Leftrightarrow x = 1/(y-1)$, this function f is injective. Since the image $(0,1) \cup (1,\infty)$ is not equal to the codomain $(0,\infty)$, the function is not surjective. Therefore, it is not a bijection.

10.

a) For any integer m $3 \mid m \Leftrightarrow m = 3n$ for some integer m. So for any integer p, $p = 3n - 1 \Leftrightarrow 3 \mid p + 1$. So a conditional definition of this set is $\{p \in \mathbb{Z} : 3 \mid p + 1\}$.

b) $0 \le \sin^2 x \le 1$ for all $x \in \mathbb{R}$. Also if $-1 \le y \le 1$ then there is $x \in \mathbb{R}$ with $\sin x = y$. If $z \in [0,1]$ and $y = \sqrt{z}$, then $z = y^2$ and if $\sin x = y$, we have $\sin^2 x = z$. So a conditional definition of this set is $\{y \in \mathbb{R} : 0 \le y \le 1\}$. A shorthand for this is just $[0,1]$, the closed interval between 0 and 1.

c)

$$\frac{x}{x+2} = y \Leftrightarrow x = xy + 2y \Leftrightarrow x = \frac{2y}{1-y}$$

which is defined for all $y \neq 1$, and is positive $\Leftrightarrow \quad 0 < y < 1$. So

$$\left\{\frac{x}{x+2} : x \in (0, \infty)\right\} = \{y \in \mathbb{R} : 0 < y < 1\}$$

So $\{y \in \mathbb{R} : 0 < y < 1\}$ is a conditional definition of this set.

11.

a) $\{n \in \mathbb{Z}_+ : 100 \mid n\} = \{100k : k \in \mathbb{Z}_+\}$.

b) $\{x \in \mathbb{R} : x < 0\} = \{1 + x^2 : x \in \mathbb{R}\}$.

12.

a) For all x and $y \in \mathbb{R}$, since both f and g are increasing,

$$x \leq y \Rightarrow g(x) \leq g(y) \Rightarrow f(g(x)) \leq f(g(y)).$$

So $f \circ g$ is increasing

b) For all x and $y \in \mathbb{R}$, since f is strictly increasing and g is strictly decreasing,

$$x < y \Rightarrow g(x) > g(y) \Rightarrow f(g(x)) > f(g(y)).$$

So $f \circ g$ is strictly decreasing. Similarly

$$x < y \Rightarrow f(x) < f(y) \Rightarrow g(f(x)) > g(f(y)).$$

So $g \circ f$ is strictly decreasing.

c) $f(x) = g(x) = x$ for all $x \in \mathbb{R}$. Clearly $f = g$ is strictly increasing but $f \cdot g(x) = x^2$ is neither increasing nor decreasing.