Solutions to Practice Problems 2

1.

a) No (because $1.5 = \frac{3}{2}$ is not an integer)

b) Yes (because 1.5 is a real number).

c) Yes (because $1 < 1.5 < 2$).

2.

a) False because $0 \in \mathbb{N}$ and it is not true that $0 > 0$.

b) True because $0 \in \mathbb{N}$ and $0 \leq 0$.

c) True because it $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$ and $n < n+1$. So given $n \in \mathbb{N}$, we can take $m = n+1$, and then $n < m$.

d) True, because given $x \in \mathbb{Q}$ with $0 < x$, if we define $y = x/2$, then $y \in \mathbb{Q}$ and $0 < y < x$.

3. Statement b) is the negation of a) and a) is the negation of b). The negation of a true statement is false and the negation of a false statement is true. This is consistent with the answers to 7, since a) is a false statement and b) is a true statement. This is not to say that any false statement is the negation of any true statement, or that any true statement is the negation of any false statement!

4.

a) Since $2.25 = \frac{9}{4} = \frac{3^2}{2^2}$, we have

$$x^2 \leq 2.25 \Leftrightarrow x^2 \leq \left(\frac{3}{2}\right)^2 \Leftrightarrow -\frac{3}{2} \leq x \leq \frac{3}{2}.$$

b) $x^2 - 3x - 4 = (x+1)(x-4)$. So

$$x^2 - 3x - 4 \geq 0 \Leftrightarrow (x+1)(x-4) \geq 0 \Leftrightarrow ((x+1 \geq 0 \wedge x-4 \geq 0) \vee (x+1 \leq 0 \wedge x-4 \leq 0))$$
$$\Leftrightarrow ((x \geq -1 \wedge x \geq 4) \vee (x \leq -1 \wedge x \leq 4)) \Leftrightarrow x \geq 4 \vee x \leq -1.$$

c)

$$\left|\frac{3x-1}{x+2}\right| < 1 \Leftrightarrow -1 < \frac{3x-1}{x+2} < 1$$
$$\Leftrightarrow ((x+2 > 0 \wedge -x-2 < 3x-1 < x+2) \vee (x+2 < 0 \wedge -(x+2) > 3x-1 > x+2))$$
$$\Leftrightarrow ((4x > -1 \wedge x < \frac{3}{2}) \vee (x < -1/4 \wedge x > \frac{3}{2}))$$
$$\Leftrightarrow -1/4 < x < \frac{3}{2}$$

because there are no x satisfying $x < -1/4 \wedge x > \frac{3}{2}$.

An **alternative method** is to square the original inequality:

$$\left|\frac{3x-1}{x+2}\right| < 1 \Leftrightarrow \frac{(3x-1)^2}{(x+2)^2} < 1 \Leftrightarrow (3x-1)^2 < (x+2)^2$$

$$\Leftrightarrow 8x^2 - 10x - 3 < 0 \Leftrightarrow (4x+1)(2x-3) \Leftrightarrow (4x+1 > 0 \wedge 2x-3 > 0) \vee (4x+1 < 0 \wedge 3x-2 < 0)$$
$$\Leftrightarrow x > \frac{3}{2} \vee x < -\frac{1}{4}$$

5. When $n = 2$,

$$2n + 1 = 5 < 8 = 2n^2$$

So $2n + 1 < 2n^2$ is true for $n = 2$. Now assume it is true for some $n \in \mathbb{N}$ with $n \geq 2$. Then

$$2(n + 1) + 1 = (2n + 1) + 2 < 2n^2 + 2 < 2n^2 + 4n + 2 = 2(n + 1)^2$$

So $2n + 1 < 2n^2 \Rightarrow 2(n + 1) + 1 < 2(n + 1)^2$ for all $n \in \mathbb{N}$ with $n \geq 2$. So $2n + 1 < 2n^2$ for all $n \in \mathbb{N}$ with $n \geq 2$.