MATH105 Feedback and Solutions 9

1. Suppose for contradiction that $x = \dfrac{p}{q}$ for p, $q \in \mathbb{Z}_+$. We can assume that p and q are the smallest possible positive integers for which this is true, and therefore that the g.c.d of p and q is 1. Then $p = xq$ and $p^3 = x^3q^3$, that is $p^3 = 4q^3$. Since $4q^3$ is even, p^3 is even, and therefore p is even. So $p = 2p_1$ for some $p_1 \in \mathbb{Z}_+$. So $8p_1^3 = 4q^3$, and $2p_1^3 = q^3$. So q^3 is even, and therefore q is also even. So both p and q is even and therefore the gcd of p and q is greater than 1, giving a contradiciton. So p and q do not exist as claimed, and x cannot be rational after all.

This was mostly well done. One point to note is that 2 is prime, but 4 is not. That is why the proof, as given above, uses divisibility by 2 and not divisibility by 4. If 2 divides a product of two integers, then 2 divides at least one of those integers. But this is not true for 4. So we can deduce, if 2 divides p^3, that 2 divides p (because $p^3 = p \times p^2$, and $p^2 = p \times p$), but if 4 divides p^3, it does not follow that 4 divides p. For example, 4 divides $8 = 2^3$, but 4 does not divide 2.

2. Write
$$a = \frac{577}{408}$$
and
$$b = a^2 - 2 = \left(\frac{577}{408}\right)^2 - 2 = \frac{332929 - 332928}{166464} = \frac{1}{166464} = \frac{1}{408^2}.$$
Then we want to take off from the square two rectangles of area $b/2$, with sides a and $\varepsilon = b/2a$. So
$$\varepsilon = \frac{b}{2a} = \frac{1}{2 \times 408^2} \times \frac{408}{577} = \frac{1}{2 \times 408 \times 577} = \frac{1}{470832}.$$
So our new approximation to $\sqrt{2}$ is
$$a - \varepsilon = \frac{577}{408} - \frac{1}{470832} = \frac{577^2 \times 2 - 1}{470832} = \frac{665857}{470832}.$$
An alternative way to work this out is to notice that
$$a - \varepsilon = a - \frac{a^2 - 2}{a} = \frac{a}{2} + \frac{1}{a}.$$
My calculator gives that the square of $a - \varepsilon$ is 2, that is, correct to more decimal places than shown. In fact
$$(a - \varepsilon^2) - 2 = a^2 - 2a\varepsilon + \varepsilon^2 - 2 = a^2 - 2 - b + \varepsilon^2 = \varepsilon^2.$$
The error is $< 10^{-13}$, so correct to 12 decimal places.

To obtain full marks on this question, you were expected to give the approximation numerically, as a fraction.

3.

a) $0 < \dfrac{1}{n^2} \leq 1 = \dfrac{1}{1}$ for all $n \in \mathbb{Z}_+$. So 1 is the maximal element. There is no minimal element, because $\dfrac{1}{(n+1)^2} < \dfrac{1}{n^2}$ for all $n \in \mathbb{Z}_+$, so $\frac{1}{n^2}$ cannot be minimal for any $n \in \mathbb{Z}_+$.

b) $\{x \in \mathbb{R} : x^2 \leq 3\} = [-\sqrt{3}, \sqrt{3}] = \{x \in \mathbb{R} : -\sqrt{3} \leq x \leq \sqrt{3}\}$. So $\sqrt{3}$ is the maximal element and $-\sqrt{3}$ is the minimal element.

c) $\{x \in \mathbb{R} : x^2 < 5\} = (-\sqrt{5}, \sqrt{5})$. This set has no maximal or minimal element because the numbers $\pm\sqrt{5}$ are not in the set, but yet for any real number $x$with $-\sqrt{5} < x < \sqrt{5}$, there are real numbers y and z with $-\sqrt{5} < y < x < z < \sqrt{5}$. For example one can take $y = (x - \sqrt{5})/2$ and $z = (x + \sqrt{5})/2$.

d) $A = \{x \in \mathbb{Q} : x^2 \leq 5\} = \mathbb{Q} \cap [-\sqrt{5}, \sqrt{5}]$ Since $\pm\sqrt{5}$ are not rational,Acan also be written as $\mathbb{Q} \cap (-\sqrt{5}, \sqrt{5})$, and again has no maximal or minimal elements because although $-\sqrt{5} < x < \sqrt{5}$ for all $x \in A$, $\pm\sqrt{5} \notin A$ and yet for any $x \in A$ there are y and $z \in A$ with $y < x < z$. For example (although I was not requiring this much detail) we can take $y = x - (5 - x^2)/6$ and $z = x + (5 - x^2)/6$.

e) $A = \mathbb{R}$ because $1 - x^2 \leq 2 \Leftrightarrow x^2 \geq -1$, and this is true for all $x \in \mathbb{R}$. So there is no maximum element and no minimum element.

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Some reasoning was required for full marks. When I write "Determine whether.. " I mean that I want some explanation.

I noticed some confusion about what to take the minimum and maximum of. *The question asks, in each case, whether the given set has a maximum* element *or a minimum* element. *Any set is written using the symbols* $\{\ :\ \}$ *where the elements of the set are written* before the colon :, *but whatever is written after the colon is needed to define the set. So this question asked if there was a minimum or maximum of what is written before the colon subject to what is written after the colon. The set in 3a) is defined* constructively, *which means that it is defined as the image of a function on some domain: in this case, the function* $f(n) = \frac{1}{n^2}$ *on the domain* $\mathbb{Z}_+$. *So what is written after the colon in this case is "*$n \in \mathbb{Z}_+$*". (The codomain of* f *is not given, but it does not need to be because the definition of the set does not depend on it.) The other sets in the question are all defined* conditionally. *Before the colon in each of these cases, we have "*$x \in \mathbb{R}$*" or "*$x \in \mathbb{Q}$*". What is written after the colon is some condition on* x. *So in each of these cases, the question asks whether there is a maximum or minimum of* x *subject to the condition. As it happens, each of the conditions in these parts of the question was of the form "*$f(x) \leq a$*" or "*$f(x) < a$ *for some function* f *and some* $a \in \mathbb{R}$. *It appears that some people thought they were supposed to find the maximum or minimum of* $f(x)$ *(where it exists) rather that of* x *itself. This may have been partly because there have recently been exercises in MATH101 on finding the mininum and maximum* values of functions *(where these exist) rather than the minimum and maximum* elements of sets, *as in this exercise.*

4. If $x \geq 1$ and $0 < \varepsilon < 1$ then

$$(x + \varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2 < x^2 + 3x\varepsilon$$

If in addition $x^2 < 3$ and $\varepsilon \leq (3 - x^2)/2$ then $x^2 + 3x\varepsilon \leq x^2 + 3 - x^2 = 3$ and hence $(x + \varepsilon)^2 < 3$.

a) $A = \{x \in \mathbb{Q} : x < 2\}$ is a Dedekind cut because for $y \in Q$, $y \geq 2 \Rightarrow y \notin A$ (which shows $A \neq \emptyset$ and $A \neq \mathbb{Q}$) and $y < x \Rightarrow y < 2 \Rightarrow y \in A$ and for $x \in A$, there is no maximal element: if $x < 2$ then $x < \frac{x+2}{2} < 2$, and $\frac{x+2}{2} \in \mathbb{Q}$.

b) In this case $A = \{x \in Q : x \leq -1\}$ which is not a Dedekind cut because a Dedekind cut is not allowed to have a maximal element and -1 is the maximal element of this set.

In the course of writing out comments I noticed that the solution given to the corresponding practice problem 8b) was incorrect. It has now been corrected.

c) $A = \{x \in Q : x^2 < 3 \vee x < 1\}$.

- $1 \in A$ and $2 \notin A$ so $A \neq \emptyset$ and $A \neq \mathbb{Q}$.
- If $x \in A$ and $y < x$ then either $y \leq 0 < 1$, in which case $y \in A$, or $0 \leq y < x$, in which case $y^2 < 3$ and again $y \in A$.
- If $x \in A$ with $x^2 < 3$, either $x < 1 \in A$ or $x \geq 1$ and we choose $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon < 3 - x^2)/3$. Then $x < x + \varepsilon$ and $x + \varepsilon \in A$. Hence x is not maximal in A for any $x \in A$ and there is no maximal element.

So A is a Dedekind cut.

d) $A = \{x \in \mathbb{Q} : x^2 < 3 \vee x > 1\}$ is not a Dedekind cut because (for example) $-2 \notin A$ but $0 \in A$

e) Again A is not a Dedekind cut because $-2 \notin A$ but $0 \in A$.

Once again, explanations are required for full marks. A Dedekind cut has been defined as satisfying four conditions. The fourth condition is "no maximal element", and the first two are simple – that a Dedekind cut is non-empty, and not all the rationals – but the third condition is also important. If a set is a Dedekind cut then all four conditions need to be checked– and it is not enough to just write out the conditions abstractly, they do need to be checked against the example. If a set is not *a Dedekind cut, then it is enough to show that just one of the conditions is* not *met. Only the sets in a) and c) turn out to be Dedekind cuts, so these are the only sets where* *all the conditions have to be checked. In order to check that $(y < x \wedge x \in A) \Rightarrow y \in A$ it is necessary to consider two cases of y: the cases $y < 1$ and $y \geq 1$ will do, and are naturally suggested by the question. The function x^2 is strictly increasing on $[1, \infty)$, so if $1 \leq y < x$ then $1 \leq y^2 < x^2$.*

Solutions to Practice Problems

5. Suppose for contradiction that $x = \dfrac{p}{q}$ for p, $q \in \mathbb{Z}_+$. We can assume that p and q are the smallest possible positive integers for which this is true, and therefore that the g.c.d of p and q is 1. Then $px = q$ and $p^3x^3 = q^3$, that is $5p^3 = q^3$. Since 2 is prime we deduce that 2 is one of the prime factors of q and hence $q = 5q_1$ for some $q_1 \in \mathbb{Z}_+$. So $5p^3 = q^3 = 5^3q_1^3$ and $p^3 = 5^2q_1^3$. So then 5 must be a prime factor of p and 5 divides both p and q the g.c.d. of p and q is not 1, giving a contradiction. So x cannot be rational after all.

6. We have $x_1 = \frac{7}{4}$ and $x_1^2 - 3 = \frac{1}{16}$. So we put

$$x_2 = x_1 - \frac{1}{16 \times 2x_1} = x_1 - \frac{4}{32 \times 7} = x_1 - \frac{1}{56} = \frac{97}{56}.$$

Then

$$x_2^2 - 3 = \frac{9409}{3136} - 3 = \frac{1}{3136}$$

So we put

$$x_3 = x_2 - \frac{1}{3136 \times 2x_2} = \frac{97}{56} - \frac{28}{3136 \times 97} = \frac{97}{56} - \frac{1}{112 \times 97}$$
$$= \frac{97^2 \times 2 - 1}{112 \times 97} = \frac{18817}{112 \times 97}.$$
$$x_3^2 - 3 = \frac{354079489}{118026496} - \frac{354079488}{118026496} = \frac{1}{118026496}.$$

7.

a) $-1 \leq -\dfrac{1}{n^3} < 0$ for all $n \in \mathbb{Z}_+$. So -1 is the mimimal element. There is no maximal element, because $-\dfrac{1}{(n+1)^3} > -\dfrac{1}{n^3}$ for all $n \in \mathbb{Z}_+$, so $\frac{1}{n^3}$ cannot be minimal for any $n \in \mathbb{Z}_+$.

b) $\{x \in \mathbb{R} : x^2 \leq 5\} = [-\sqrt{5}, \sqrt{5}] = \{x \in \mathbb{R} : -\sqrt{5} \leq x \leq \sqrt{5}\}$. So $\sqrt{5}$ is the maximal element and $-\sqrt{5}$ is the minimal element.

c) $\{x \in \mathbb{R} : x^2 < 3\} = (-\sqrt{3}, \sqrt{3})$. This set has no maximal or minimal element because the numbers $\pm\sqrt{3}$ are not in the set, but yet for any real number x with $-\sqrt{3} < x < \sqrt{3}$, there are real numbers y and z with $-\sqrt{3} < y < x < z < \sqrt{3}$. For example one can take $y = (x - \sqrt{3})/2$ and $z = (x + \sqrt{3})/2$.

d) $A = \{x \in \mathbb{Q} : x^2 \leq 3\} = \mathbb{Q} \cap [-\sqrt{3}, \sqrt{3}]$ Since $\pm\sqrt{3}$ are not rational,Acan also be written as $\mathbb{Q} \cap (-\sqrt{3}, \sqrt{3})$, and again has no maximal or minimal elements because although $-\sqrt{3} < x < \sqrt{3}$ for all $x \in A$, $\pm\sqrt{3} \notin A$ and yet for any $x \in A$ there are y and $z \in A$ with $y < x < z$. For example (although this much detail is not required) we can take $y = x - (3 - x^2)/4$ and $z = x + (3 - x^2)/4$.

e) $A = \mathbb{R}$ because $2 - x^2 \leq 3 \Leftrightarrow x^2 \geq -1$, and this is true for all $x \in \mathbb{R}$. So there is no maximum element and no minimum element.

8. If $x \geq 2$ and $0 < \varepsilon < 1$ then

$$(x + \varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2 < x^2 + 3x\varepsilon$$

If in addition $x^2 < 5$ and $\varepsilon < (5 - x^2)/3$ then $x^2 + 3x\varepsilon \leq x^2 + 5 - x^2 = 5$ and hence $(x + \varepsilon)^2 < 5$.

a) $A = \{x \in \mathbb{Q} : x < 3\}$ is a Dedekind cut because for $y \in Q$, $y \geq 2 \Rightarrow y \notin A$ (which shows $A \neq \emptyset$ and $A \neq \mathbb{Q}$) and $y < x \Rightarrow y < 2 \Rightarrow y \in A$ and for $x \in A$, there is no maximal element: if $x < 2$ then $x < \frac{x+2}{2} < 2$, and $\frac{x+2}{2} \in \mathbb{Q}$.

b) In this case $A = \{x \in Q : x \leq -2\}$ which is not a Dedekind cut because -2 is a maximal element.

c) $A = \{x \in Q : x^2 < 5 \vee x < 1\}$.

- $2 \in A$ and $3 \notin A$. So $A \neq \emptyset$ and $A \neq \mathbb{Q}$.
- If $x \in A$ and $y < x$ then either $y \leq 0 < 1$, in which case $y \in A$, or $0 \leq y < x$, in which case $y^2 < 5$ and again $y \in A$.
- If $x \in A$ with $x^2 < 5$, either $x < 1 \in A$ or $x \geq 1$. If x is maximal we must have $x \geq 2$, and we choose $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon = (5 - x^2)/4$. Then $x < x + \varepsilon$ and $x + \varepsilon \in A$. Hence x is not maximal in A for any $x \in A$ and there is no maximal element.

So A is a Dedekind cut.

d) $A = \{x \in \mathbb{Q} : x^2 < 5 \vee x > 1\}$ is not a Dedekind cut because (for example) $-3 \notin A$ but $0 \in A$

e) Again A is not a Dedekind cut because $-3 \notin A$ but $0 \in A$.

9. *Base case* If $A \subset \mathbb{R}$ has one element a then a is both a maximal and a minimal element of A.

Inductive step Now suppose inductively that any set with n elements has both a maximal element and a minimal element. Let A be a set with $n + 1$ elements. Then there is a bijection $f : \{k \in \mathbb{Z}_+ : k \leq n + 1\} \to A$. Let $B = \{f(k) : k \leq n\}$. Then by the inductive hypothesis B has a maximal element b_1 and a minimal element b_2, where $b_1 = f(i)$ for some $i \leq n$ and $b_2 = f(j)$ for some $j \leq n$. Then $A = B \cup \{f(n + 1)$. There are three possibilities.

- $f(n+1) < b_2$. Then $f(n+1)$ is the minimum element of A and $b_1 = f(i)$ is the maximum element.
- $b_1 < f(n+1)$. Then $b_2 = f(j)$ is the minimum element of A and $f(n+1)$ is the maximum element.
- $n \geq 2$ and $b_2 < f(n+1) < b_1$. Then $b_2 = f(j)$ is the minimum element of A and b_1 is the maximum element.

So by induction a set with n elements has both a maximal and a minimal element for all $n \in \mathbb{Z}_+$.