MATH105 Feedback and Solutions 8

1.

a) (i) $m - m = 0$, so $10|m - m$ and $m \sim m$ for all $m \in \mathbb{Z}$ and $\sim$ is *reflexive*.

Note that $10|0$ *means* 10 divides 0, *which is* true. *More than one person decided it meant* 0 divides 10, *which is of course* false. *Such people naturally then decided that the relation was* not reflexive, *but* 10 divides 0 *is* true *and therefore the relation* is *reflexive.*

(ii) $m - n = 10k \Leftrightarrow n - m = 10(-k)$, and $k \in \mathbb{Z} \Leftrightarrow -k \in \mathbb{Z}$. So $10|m - n \Leftrightarrow 10|n - m$ and $m \sim n \Leftrightarrow n \sim m$ and $\sim$ is *symmetric*

(iii)

$$m - n = 10k_1 \wedge n - p = 10k_2 \Rightarrow m - p = 10(k_1 + k_2),$$

and

$$k_1 \in \mathbb{Z} \wedge k_2 \in \mathbb{Z} \Rightarrow k_1 + k_2 \in \mathbb{Z}$$

So

$$m \sim n \wedge n \sim p \Rightarrow m \sim p$$

and $\sim$ is *transitive*

Hence $\sim$ is an equivalence relation. The equivalence classes are the sets

$$\{k + 10m : m \in \mathbb{Z}\}$$

for $k \in \mathbb{Z}$, with $0 \leq k \leq 9$. Every integer is in exactly one of these sets. So there are exactly 10 equivalence classes.

The equivalence classes are sets, *so should be written down as sets, that is, enclosed in curly brackets. It is OK to write the sets informally, such as* $\{\cdots - 10, 0, 10, 20 \cdots\}$. *The definitions given above are constructive. Conditional definitions of the same sets are given by*

$$\{n \in \mathbb{Z} : 10|(n - k)\}$$

for $k \in \mathbb{Z}$, *with* $0 \leq k \leq 9$.

b) If $m = 2$ then $2 \times 2 \neq 1$ and so it is not true that $2 \sim 2$. So $\sim$ is not *reflexive*, that is, it is *not* true that $m \sim m$ for all $m \in \mathbb{Z}$. and $\sim$ is not an equivalence relation.

Reflexivity has to hold for all integers. It is true that $1 \sim 1$ *and* $-1 \sim -1$, *but that is not enough. Since this relation is not reflexive, it cannot be an equivalence relation, and there is no need to check symmetry and transitivity as well.(Actually the relation is both symmetric and transitive, as some people worked out, but since it is not reflexive it cannot be an equivalence relation.)*

c) (i) $m - m = 0 = 0 \times k^2$ for any integer k, so $m \sim m$ for all $m \in \mathbb{Z}$ and $\sim$ is *reflexive.*

(ii) $m - n = rk^2 \Leftrightarrow n - m = (-r) \cdot k^2$, So $m \sim n \Leftrightarrow n \sim m$ and $\sim$ is *symmetric*

(iii) If $m = 0$ and $n = 4$ and $p = 13$, then $2^2|0 - 4$ and $3^2|4 - 13$. So $0 \sim 4$ and $4 \sim 13$. But 13 is prime and there is no integer $k \geq 2$ such that $k^2|13$. So $\sim$ is *not transitive.*

So $\sim$ is not an equivalence relation.

Despite the italics in the question, most people did not take in that the integer k *in the definition of* $m \sim n$ depends on m and n. *If you spot straight away that this relation is not transitive, then there is no need to check reflexivity and symmetry as well.*

2.

a) (i) $|A| = |A|$ for all $A \subset Y$. So $\sim$ is *reflexive*

(ii) For A, $B \subset Y$, $|A| = |B| \Leftrightarrow |B| = |A|$. So $\sim$ is *symmetric*

(iii) If A, B and $C \subset Y$, then

$$|A| = |B| \wedge |B| = |C| \Rightarrow |A| = |C|.$$

So $\sim$ is *transitive*

Hence $\sim$ is an equivalence relation.

b) If $Y = \{1,2,3,4\}$ then there are 5 equivalence classes:

$$\begin{array}{l} \{\emptyset\}, \\ \{\{1\},\{2\},\{3\},\{4\}\}, \\ \{\{1,2\},\{2,3\},\{3,4\},\{1,3\},\{2,4\},\{1,4\},\}. \\ \{\{1,2,3,\},\{1,3,4\}.\{1,2,4\},\{2,3,4\}\}, \\ \{\{1,2,3,4\}\}. \end{array}$$

The elements of X are subsets of Y *and the equivalence classes of* $\sim$ *are* subsets of X, *which makes them* sets of subsets of Y *— hence the nested curly brackets in the solution given above. I accepted solutions where people simply wrote the elements of the equivalence classes on 5 different lines The notation 2^Y for the set of subsets of Y confused many people. Quite a few people did write down all the subsets of $\{1,2,3,4\}$, and that was worth doing, but does not answer the question asked. It is a standard notation for the set of subsets of a set Y. One reason for the notation is that, if Y is finite then the number of subsets of Y is $2^{|Y|}$, where (as usual) $|Y|$ is the number of elements of Y. So in this example, since $|Y| = 4$, the number of subsets of Y is $2^4 = 16$.*

c) If $|Y| = n$ then the number of equivalence classes of $\sim$ is $n+1$, because two subsets of Y are equivalent $\Leftrightarrow$ they have the same number r of elements for some r, with $0 \le r \le n$. The number of subsets of Y with r elements is

$$\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}.$$

So these are the sizes of the equivalence classes, for $0 \le r \le n$.

Some solutions mentioned a connection with Pascal's triangle which is useful information but you need to say a bit more to make the answer complete. The k'th entry of the n'th row (starting from $k = 0$) is the number of k-element subsets of a set with n elements.

3.

a) The least n is the l.c.m. of $4 = 2^2$ and $15 = 3 \times 5$ and $35 = 5 \times 7$. So $n = 2^2 \times 3 \times 5 \times 7 = 420$.

b) *There is more than one correct answer: in fact there are infinitely many.*

Since $n = 420 = 12 \times 35 = 15 \times 28 = 35 \times 12$, we have $n_1 = 35$ and $n_2 = 28$. So the g.c.d. d of n_1 and n_2 is 7, and

$$\frac{a}{4} + \frac{b}{15} + \frac{c}{35} = \frac{105a + 28b + 12c}{420} = \frac{7(35a + 4b) + 12c}{420}.$$

So it suffices to find a_1 and b_1 such that $15a_1 + 4b_1 = 1$ and e such that $7e + 12c = 1$ and and then take $a = a_1 e$ and $b = b_1 e$. In fact we then have

$$\frac{a_1 e}{4} + \frac{b_1 e}{15} = \frac{e(15a_1 + 4b_1)}{60} = \frac{e}{60}$$

and hence

$$\frac{a_1 e}{12} + \frac{b_1 e}{15} + \frac{c}{35} = \frac{e}{60} + \frac{c}{35} = \frac{7e + 12c}{420} = \frac{1}{420}.$$

One correct answer By inspection we can take $a_1 = -1$ and $b_1 = 4$ and $e = -5$ and $c = 3$. Then $a = 5$ and $b = -20$. So

$$\frac{1}{420} = \frac{5}{4} + \frac{-20}{15} + \frac{3}{35} = \frac{5}{4} - \frac{4}{3} + \frac{3}{35} = -\frac{1}{12} + \frac{3}{35} = \frac{1}{420}$$

4.

a) Suppose for contradiction that there are $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$ such that

$$\frac{1}{3} + \frac{2}{5}\sqrt{2} = \frac{p}{q}$$

Then

$$\sqrt{2} = \frac{5p}{2q} - \frac{5}{6} = \frac{15p - 5q}{6q} \in \mathbb{Q}$$

which is a contradiction.

b) Suppose for contradiction that $x \in \mathbb{Q}$ with $a + b\sqrt{2} = x$. Since $b \neq 0$, $b^{-1} \in \mathbb{Q}$ exists and $\sqrt{2} = (x - a) \cdot b^{-1} \in \mathbb{Q}$ which is again a contradiction.

c) If $2^{1/4} = x \in \mathbb{Q}$ then $\sqrt{2} = x^2 \in \mathbb{Q}$, which is a contradiction.

Most solutions that I saw did use proof by contradiction and I think this is the clearest way to do it.

Solutions to Practice Problems

5.

a) (i) $x - x = 0 \in \mathbb{Z}$, so $x \sim x$ for all $x \in \mathbb{R}$ and $\sim$ is *reflexive.*

(ii) $x - y \in \mathbb{Z} \Leftrightarrow y - x \in \mathbb{Z}$). So, for all $x,\, y \in \mathbb{R}$, $x \sim y \Leftrightarrow y \sim x$, and $\sim$ is *symmetric*

(iii)
$$x - y \in \mathbb{Z} \wedge y - z \in \mathbb{Z} \Rightarrow x - z = (x - y) + (y - z) = x - z \in \mathbb{Z}.$$

So, for all $x,\, y,\, z \in \mathbb{Z}$,
$$x \sim y \wedge y \sim z \Rightarrow x \sim z,$$
and $\sim$ is *transitive.*

Hence $\sim$ is an equivalence relation.

b) If $x = 1$ and $y = 0$ then $x - y = 1 \in \mathbb{N}$ and hence $1 \sim 0$. But $0 - 1 = -1 \notin \mathbb{N}$ and hence it is not true that $0 \sim 1$. So $\sim$ is not symmetric, and is not an equivalence relation on $\mathbb{R}$.

c) If $m = 2$ and $n = 1$ then $m/n = 2 \in \mathbb{Z}_+$ and so $2 \sim 1$. But $n/m = 1/2 \notin \mathbb{Z}_+$ and so it is not true that $1 \sim 2$. So $\sim$ is not symmetric, and is not an equivalence relation on $\mathbb{Z}_+$.

6.

(i) For any $m,\, n \in \mathbb{Z}$, $m - m = 0$ and $n - n = 0$, and 0 is even. So $(m, n) \sim (m, n)$ and $\sim$ is *reflexive.*

(ii) For any $m_1,\, n_1,\, m_2,\, n_2 \in \mathbb{Z}$,

$$\begin{aligned}(m_1, n_1) \sim (m_2, n_2) \quad &\Rightarrow \quad (m_1 - m_2 \text{ even } \wedge n_1 - n_2 \text{ even }) \quad \Rightarrow \quad (m_2 - m_1 \text{ even } \wedge n_2 - n_1 \text{ even }) \\ &\Rightarrow \quad (m_2, n_2) \sim (m_1, n_1).\end{aligned}$$

So $\sim$ is *symmetric*

(iii) For any $m_1,\, n_1,\, m_2,\, n_2,\, m_3,\, n_3 \in \mathbb{Z}$,

$$\begin{aligned}((m_1, n_1) \sim (m_2, n_2) \wedge (m_2, n_2) \sim (m_3, n_3)) \quad &\Rightarrow \quad (m_1 - m_2 \text{ even } \wedge n_1 - n_2 \text{ even } \wedge m_2 - m_3 \text{ even } \wedge n_2 - n_3 \text{even}) \\ &\Rightarrow \quad (m_2 - m_1 + m_3 - m_2 \text{ even } \wedge n_2 - n_1 + n_3 - n_2 \text{ even }) \\ &\Rightarrow \quad (m_3 - m_1 \text{ even } \wedge n_3 - n_1 \text{ even }) \quad \Rightarrow \quad (m_1, n_1) \sim (m_3, n_3).\end{aligned}$$

So $\sim$ is *transitive.*

So $\sim$ is an equivalence relation. If we consider the four vectors

$$(0, 0), \quad (1, 0), \quad (0, 1), \quad (1, 1),$$

then these are in four different equivalence classes. Denoting the equivalence class of a vector (m, n) by $[(m, n)]$, we have

$$\begin{aligned}[(0, 0)] &= \{(2k, 2\ell) : k, \ell \in \mathbb{Z}\}, \\ [(1, 0)] &= \{(2k + 1, 2\ell) : k, \ell \in \mathbb{Z}\}, \\ [(0, 1)] &= \{(2k, 2\ell + 1) : k, \ell \in \mathbb{Z}\}, \\ [(1, 1)] &= \{(2k + 1, 2\ell + 1) : k, \ell \in \mathbb{Z}\}.\end{aligned}$$

7.

a) We have $15 = 3 \times 5$ and $35 = 5 \times 7$ and 11 is prime. So the l.c.m. n of 11, 15 and 35 is $11 \times 3 \times 5 \times 7 = 11 \times 105 = 1155$.

b) The l.c.m. of 15 and 35 is $15 \times 7 = 3 \times 35 = 105$. So we have

$$\frac{b_1}{15} + \frac{c_1}{35} = \frac{7b_1 + 3c_1}{105}.$$

One solution to $7b_1 + 3c_1 = 1$ is $b_1 = 1$ and $c_1 = -2$. Then

$$\frac{a}{11} + \frac{e}{105} = \frac{105a + 11e}{1155} = \frac{1}{1155} \Leftrightarrow 105 + 11e = 1.$$

To solve this using the Euclidean algorithm,

$$\begin{array}{cc|c}1 & 0 & 105\\ 0 & 1 & 11\end{array} \; \begin{array}{c} R_1 - 9R_2 \\ \rightarrow \end{array} \; \begin{array}{cc|c}1 & -9 & 6\\ 0 & 1 & 11\end{array} \; \begin{array}{c} \rightarrow \\ R_2 - R_1 \end{array} \; \begin{array}{cc|c}1 & -9 & 6\\ -1 & 10 & 5\end{array} \; \begin{array}{c} R_1 - R_2 \\ \rightarrow \end{array} \; \begin{array}{cc|c}2 & -19 & 1\\ -1 & 10 & 5\end{array}$$

So we can take $a = 2$ and $e = -19$ which gives

$$a = 2, \quad b = -19, \quad c = 38.$$

Thus

$$\frac{2}{11} - \frac{19}{15} + \frac{38}{35} = \frac{1}{1155}.$$

Of course there are many other solutions.

8.

a) Suppose for contradiction that there are $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$ such that

$$\frac{1}{7} + \frac{5}{4}\sqrt{3} = \frac{p}{q}$$

Then

$$\sqrt{3} = \frac{4p}{5q} - \frac{4}{35} = \frac{28p - 20q}{35q} \in \mathbb{Q}$$

which is a contradiction.

b) Suppose for contradiction that $x \in \mathbb{Q}$ with $a + b\sqrt{3} = x$. Since $b \neq 0$, $b^{-1} \in \mathbb{Q}$ exists and $\sqrt{3} = (x - a) \cdot b^{-1} \in \mathbb{Q}$ which is again a contradiction.

c) If $3^{1/6} = x \in \mathbb{Q}$ then $\sqrt{3} = x^3 \in \mathbb{Q}$, which is a contradiction.

9. $3^0 + (-1)^0 = 1 + 1 = 2 = 3^1 + (-1)^1$, the formula $x_k = 3^k + (-1)^k$ holds for $k = 0$ and $k = 1$.
Now suppose the formula holds for $k \leq n$. Then

$$x_{n+1} = 2x_n + 3x_{n-1} = 2(3^n + (-1)^n) + 3(3^{n-1} + (-1)^{n-1}) = 2 \cdot 3^n + 3^n + 2 \cdot (-1)^n - 3 \cdot (-1)^n$$
$$= 3^{n+1} + (-1)^{n+1}$$

So if the formula holds for $k \leq n$ then it also holds for $k = n + 1$, and hence for $k = n + 1$.
So by induction $x_n = 3^n + (-1)^n$ for all n.

As noted in the question, we need two *bases cases, and the most natural ones to take are $n = 0$ and $n = 1$. Also, as noted in the question, we need to assume* at least the two cases *$k = n - 1$ and $k = n$ in order to get the case $k = n + 1$. In the solution above I have chosen to make the inductive hypothesis "Assume true for $k \leq n$", to deduce the case for $k = n + 1$, which means that "True for $k \leq n$ implies "True for $k \leq n + 1$".*