MATH105 Feedback and Solutions 6

1.

Remember that intervals are *sets of real numbers* and any interval $[a, b]$ (or (a, b) r $[a, b)$ or $(a, b]$) with $a < b$ contains infinitely many real numbers.

a) $(1, \infty) \cap [0, 2] = \{x : x > 1 \wedge 0 \leq x \leq 2\} = \{x : 1 < x \leq 2\} = (1, 2]$

b) $(1, 3) \cup [0, 2] = \{x : 1 < x < 3 \vee 0 \leq x \leq 2\} = \{x : 0 \leq x < 3\} = [0, 3)$

c) $([3, 5] \cup [0, 4]) \setminus [2, 6] = [0, 5] \setminus [2, 6] = \{x : 0 \leq x \leq 5 \wedge \neg (2 \leq x \leq 6)\} = \{x : 0 \leq x < 2\} = [0, 2)$

d) $[3, 5] \cup ([0, 4] \setminus [2, 6]) = [3, 5] \cup [0, 2)$.

There is evidence for many people being unaware, at least some of the time, that intervals are sets of real numbers, *not integers – although of course the set of integers is included in the set of real numbers. This caused some mistakes in this question. I had quite a few answers saying "$[0, 2) \cup [3, 5] = [0, 5]$". The correct answer is $[0, 2) \cup [3, 5]$ but this is* not *equal to $[0, 5]$ and I subtracted a mark when it was said that it was, because this is important. Of course, 2 is not in $[0, 2) \cup [3, 5]$ but that is not all. There are* infinitely many *real numbers between 2 and 3. Any infinite decimal $2.a_1a_2\cdots$ where not all the a_i are 0 and not all the a_i are 9 is a real number between 2 and 3. Of course this also includes all rational numbers p/q with $2q < p < 3q$ for q, $p \in \mathbb{Z}_+$ — but not just these.*

2.

a) For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$, the image is $[0, \infty)$ since if $x \geq 0$ then $f(x) = x$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$

b) For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 1$ the image is $\mathbb{R}$ because for $y \in \mathbb{R}$,

$$f(x) = y \;\Leftrightarrow\; 2x + 1 = y \;\Leftrightarrow\; x = \frac{y - 1}{2}$$

c) For $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \ln x$ the image is $\mathbb{R}$ because $ln(e^x) = x$

d) For $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \dfrac{1}{1 + x}$, we have $1 + x > 1$ for all $x > 0$ and hence $0 < \dfrac{1}{1 + x} < 1$ for all $x > 1$. So the image is contained in $(0, 1)$. To see that if is precisely this, we see that for $0 < y < 1$,

$$\frac{1}{1 + x} = y \Leftrightarrow 1 = y + xy \Leftrightarrow 1 - y = xy \Leftrightarrow x = \frac{1 - y}{y}. \qquad (1)$$

Since $\dfrac{1 - y}{y} > 0$ for any y with $0 < y < 1$, the image is indeed $(0, 1)$.

I accepted answers which justified the answer with a graph of the function $\dfrac{1}{1 + x}$ for $x > 0$, provided the graph was clear and the asymptotes were also clear. It is also possible to use the Intermediate Value theorem. To make a completely correct argument using the IVT, note that f is continuous and strictly decreasing on $[0, \infty)$, with $f(1) = 1$ and $\lim_{x\to\infty} f(x) = 0$. So $\mathrm{Imf}(f) \subset (0, 1)$, and, by the Intermediate Value Theorem, for any $y \in (0, 1)$ there is $x \in (0, \infty)$ with $f(x) = y$ (which is actually unique, because f is strictly decreasing). So $\mathrm{Im}(f) = (0, 1)$.

3.

(i) The map in 2a) is not injective because $|x| = |-x|$ for all $x \in \mathbb{R}$ and $x \neq -x$ except when $x = 0$. The maps in b) and c) have inverses and are therefore injective. The map in d) is also injective because an inverse map is defined on the image: $f(x) = y \Leftrightarrow x = \frac{1-y}{y}$.

(ii) The maps in a) and d) are not surjective because in each case the image is a proper subset of the codomain. The maps in b) and c) are surjective because the image is the same as the codomain in each case.

(iii) A map is bijective if and only if it is both injective and surjective. So the maps in b) and c) are bijective and the maps in a) and d) are not.

4. Let $f : X \to Y$ and $g : Y \to Z$ be maps.

a) To show that $g \circ f : X \to Z$ is surjective, we need to show that for any $z \in Z$, there is $x \in X$ such that $g \circ f(x) = z$. So fix $z \in Z$. Since $g : Y \to Z$ is surjective, there is $y \in Y$ such that $g(y) = z$. Since $f : X \to Y$ is surjective, there is $x \in X$ such that $f(x) = y$. Then $g \circ f(x) = g(f(x)) = g(y) = z$. So $g \circ f$ is surjective.

The notation $f(X)$ for $\mathrm{Im}(f) = \{f(x) : x \in X\}$ is an accepted notation. Similarly one can write $g(Y)$ for $\mathrm{Im}(g)$. So f being surjective means $f(X) = Y$ and g being surjective means $g(Y) = Z$. So one way of writing the answer to this question is to write: "since f and g are surjective, $(g \circ f)(X) = g(f(X)) = g(Y) = Z$". However, one has to be careful to distinguish between the sets X, Y and Z and the various elements of them, which it is natural to denote by x, y and z. So it is not enout to write "$g(f(x)) = g(y) = z$" with little or no comment. That shows something of the right idea but it is not enough.

b) Suppose that $g \circ f : X \to Z$ is surjective. We need to show that for any $z \in Z$ there is $y \in Y$ such that $g(y) = z$. So fix $z \in Z$. There is $x \in X$ such that $(g \circ f)(x) = z$, that is, $g(f(x)) = z$. Put $y = f(x)$. Then $g(y) = z$. So g is surjective.

Alternatively one can use the fact that $\mathrm{Im}(g \circ f) \subset \mathrm{Im}(g)$. I proved this in lectures. If $g \circ f$ is surjective then it follows that

$$Z = \mathrm{Im}(g \circ f) \subset \mathrm{Im}(g) \circ Z.$$

So $Z \subset \mathrm{Im}(g) \subset Z$ and it follows that $\mathrm{Im}(g) = Z$, that is, g is surjective.

c) A very simple example is as follows $g : \mathbb{R} \to [0, \infty)$ defined by $g(x) = x^2$ and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then $g \circ f : \mathbb{R} \to [0, \infty)$ is defined by $g \circ f(x) = x^4$. Then $g \circ f$ is surjective and g is surjective but f is not.

This may seem like cheating because f and g are given by the same formula but are regarded as different maps because they have different codomains. But something like that will be true for any example because in order for $g \circ f$ to be surjective, it has to be the case that $g(\mathrm{Im}(f)) = Z$. So in order for f to be not surjective, f must have a codomain different from $\mathrm{Im}(f)$.

In order to define a function, the domain and codomain have to be specified, which some people failed to do. Also, if the composition $g \circ f$ is defined, then the codomain of f has to equal the domain of g. By part b), in order to make an example, g must be surjective – but not injective, because otherwise g is a bijection and $f = g^{-1} \circ (g \circ f)$ must be surjective. Also it is clear that to make an example, g must map $\mathrm{Im}(f)$ to $\mathrm{Im}(g)$.

5. For conditional definitions,

a) $\{5n + 1 : n \in \mathbb{Z}\} = \{m \in Z : 5|m - 1\}$

b) $\{x^2 - 1 : x \in \mathbb{R}\} = \{y \in \mathbb{R} : y \geq -1\}$, because any real number ≥ -1 can be written in the form $x^2 - 1$ for some $x \in \mathbb{R}$, and $x^2 - 1 \geq -1$ for all $x \in \mathbb{R}$.

c) If $x > 0$ then $0 < \dfrac{x}{x+1} < 1$. Also,

$$y = \frac{x}{x+1} \quad \Leftrightarrow \quad x = \frac{y}{1-y}$$

and if $0 < y < 1$ then $\dfrac{y}{1-y} > 0$. So $\left\{\dfrac{x}{x+1} : x \in (0,\infty)\right\} = \{y \in \mathbb{R} : 0 < y < 1\}$.

Here you were given sets defined constructively *and were asked to define them* conditionally. *That means putting some condition on the elements of the set such as "$y \geq -1$". Remember that curly brackets mean "the set of" and a colon means "such that". So, for example $\{y \in \mathbb{R} : y \geq -1\}$ means "the set of all real numbers y such that $y \geq -1$." Once again, there were signs of p[eople forgetting that there are many more real numbers than integers. So, for instance, there are many real numbers in $(0,\infty)$ that are not integers, and there are many real numbers in $(0,1)$ which are not rational.*

6. For constructional definitions:

a) $\{n \in \mathbb{Z}_+ : 2 \not| n\} = \{2m+1 : m \in \mathbb{N}\}$ because this set is the set of odd positive integers.Alternatively, we can write this as $\{2m-1 : m \in \mathbb{Z}_+\}$.

b) $\{x \in \mathbb{R} : x \leq 0\} = \{-x^2 : x \in \mathbb{R}\}$.

c) $\{n \in \mathbb{Z}_+ : p \not| n$ for any prime $p \geq 3\} = \{2^n : n \in \mathbb{N}\}$ because the conditions imply that 2 is the only prime divisor of numbers in the set.

This is an example of a set where there is a constructional definition which is simpler than any conditional definition that I can think of.

Solutions to Practice Problems

7.

a) $[1,3) \cap (2,4] = \{x \in \mathbb{R} : 1 \leq x < 3 \wedge 2 < x \leq 4\} = (2,3)$.

b) $[1,3) \cup (2,4] = \{x \in \mathbb{R} : 1 \leq x < 3 \vee 2 < x \leq 4\} = [1,4]$.

c) $([2,5] \cup [1,4]) \setminus (0,3) = \{x \in \mathbb{R} : (2 \leq x \leq 5 \vee 1 \leq x \leq 4) \wedge \neg\, (0 < x < 3) = \{x \in \mathbb{R} : (1 \leq x \leq 5) \wedge (x \leq 0 \vee x \geq 3\} = [3,5]$.

d) $[2,5] \cup ([1,4] \setminus (3,4) = \{x \in \mathbb{R} : (2 \leq x \leq 5 \vee 1 \leq x \leq 4) \wedge \neg\, (3 < x < 4) = \{x \in \mathbb{R} : (1 \leq x \leq 5) \wedge (x \leq 3 \vee x \geq 4\} = [1,3] \cup [4,5]$.

8.

a) $x^4 = (x^2)^2 \geq 0$ for all $x \in \mathbb{R}$, so the image of f is contained in $[0,\infty)$. Conversely, if $y \geq 0$ then $y^{1/4} = \sqrt{\sqrt{y}}$ exists and $(y^{1/4})^4 = y$. So the image of f is $[0,\infty)$.

b) $f(x) = y \Leftrightarrow x - 2 = y \Leftrightarrow x = y + 2$. So $\mathrm{Im}(f) = \mathbb{R}$.

c) Since $e^x > 0$ for all $x \in \mathbb{R}$, the image of f is contained in $(0,\infty)$. For any $y > 0$,

$$e^x = y \Leftrightarrow x = \ln y.$$

So $\mathrm{Im}(f) = (0,\infty)$.

d)

$$y = 1 + \frac{1}{x} \Leftrightarrow x = \frac{1}{y-1}$$

Since $\dfrac{1}{y-1}$ is defined for $y \neq 1$, the image of f is $(0,1) \cup (1,\infty) = \{y \in (0,\infty : y \neq 1\}$.

9.

a) This function f is not injective because, for example, $1^4 = (-1)^4$. It is not surjective because the image $[0,\infty)$ is not equal to the codomain $\mathbb{R}$.

b) This function f is injective and surjective because $f(x) = y \Leftrightarrow x = y + 2$. So for each y there is at most one x with $f(x) = y$ (injective) and at least one x with $f(x) = y$ (surjective). Therefore it is also a bijection, since it is both injective and surjective.

c) Since $f(x) = y \Leftrightarrow x = \ln y$, this function f is injective. Since the image $(0,\infty)$ is not equal to the codomain $\mathbb{R}$, the function is not surjective. Therefore, it is not a bijection.

d)

e) Since $f(x) = y \Leftrightarrow x = 1/(y-1)$, this function f is injective. Since the image $(0,1) \cup (1,\infty)$ is not equal to the codomain $(0,\infty)$, the function is not surjective. Therefore, it is not a bijection.

10.

a) For any integer m $3 \mid m \Leftrightarrow m = 3n$ for some integer m. So for any integer p, $p = 3n - 1 \Leftrightarrow 3 \mid p + 1$. So a conditional definition of this set is $\{p \in \mathbb{Z} : 3 \mid p + 1\}$.

b) $0 \leq \sin^2 x \leq 1$ for all $x \in \mathbb{R}$. Also if $-1 \leq y \leq 1$ then there is $x \in \mathbb{R}$ with $\sin x = y$. If $z \in [0,1]$ and $y = \sqrt{z}$, then $z = y^2$ and if $\sin x = y$, we have $\sin^2 x = z$. So a conditional definition of this set is $\{y \in \mathbb{R} : 0 \leq y \leq 1\}$. A shorthand for this is just $[0,1]$, the closed interval between 0 and 1.

c)

$$\frac{x}{x+2} = y \Leftrightarrow x = xy + 2y \Leftrightarrow x = \frac{2y}{1-y}$$

which is defined for all $y \neq 1$, and is positive $\Leftrightarrow \quad 0 < y < 1$. So

$$\left\{\frac{x}{x+2} : x \in (0,\infty)\right\} = \{y \in \mathbb{R} : 0 < y < 1\}$$

So $\{y \in \mathbb{R} : 0 < y < 1\}$ is a conditional definition of this set.

11.

a) $\{n \in \mathbb{Z}_+ : 100 \mid n\} = \{100k : k \in \mathbb{Z}_+\}$.

b) $\{x \in \mathbb{R} : x < 0\} = \{1 + x^2 : x \in \mathbb{R}\}$.

12.

a) For all x and $y \in \mathbb{R}$, since both f and g are increasing,

$$x \leq y \Rightarrow g(x) \leq g(y) \Rightarrow f(g(x)) \leq f(g(y)).$$

So $f \circ g$ is increasing

b) For all x and $y \in \mathbb{R}$, since f is strictly increasing and g is strictly decreasing,

$$x < y \Rightarrow g(x) > g(y) \Rightarrow f(g(x)) > f(g(y)).$$

So $f \circ g$ is strictly decreasing. Similarly

$$x < y \Rightarrow f(x) < f(y) \Rightarrow g(f(x)) > g(f(y)).$$

So $g \circ f$ is strictly decreasing.

c) $f(x) = g(x) = x$ for all $x \in \mathbb{R}$. Clearly $f = g$ is strictly increasing but $f \cdot g(x) = x^2$ is neither increasing nor decreasing.