MATH105 Feedback and Solutions 4

1. *Note that the answers for a and b in each part are not unique: there are infinitely many correct answers for a and b. The array notation has been used throughout this solution sheet. The other way of writing — $r_{i-1} = q_{i+1}r_i + r_{i+1}$ – was the first way that was shown for writing out the Euclidean algorithm, and is also valid.*

a) $m = 28$, $n = 105$.

$$\left.\begin{array}{cc|c} 1 & 0 & 28 \\ 0 & 1 & 105 \end{array}\right. \xrightarrow{R_2 - 3R_1} \left.\begin{array}{cc|c} 1 & 0 & 28 \\ -3 & 1 & 21 \end{array}\right. \xrightarrow{R_1 - R_2} \left.\begin{array}{cc|c} 4 & -1 & 7 \\ -3 & 1 & 21 \end{array}\right. \xrightarrow{R_2 - 3R_1} \left.\begin{array}{cc|c} 4 & -1 & 7 \\ -15 & 4 & 0 \end{array}\right.$$

So (i) $d = 7$, (ii) $a = 4$ and $b = -1$, that is, $4 \times 28 - 1 \times 105 = 7$ and (iii) $m_1 = 4$ and $n_1 = 15$, that is, $28 = 4 \times 7$ and $105 = 15 \times 7$.

b) $m = 213$, $n = 741$.

$$\left.\begin{array}{cc|c} 1 & 0 & 213 \\ 0 & 1 & 741 \end{array}\right. \xrightarrow{R_2 - 3R_1} \left.\begin{array}{cc|c} 1 & 0 & 213 \\ -3 & 1 & 102 \end{array}\right. \xrightarrow{R_1 - 2R_2} \left.\begin{array}{cc|c} 7 & -2 & 9 \\ -3 & 1 & 102 \end{array}\right. \xrightarrow{R_2 - 11R_1} \left.\begin{array}{cc|c} 7 & -2 & 9 \\ -80 & 23 & 3 \end{array}\right. \xrightarrow{R_1 - 3R_2} \left.\begin{array}{cc|c} 247 & -71 & 0 \\ -80 & 23 & 3 \end{array}\right.$$

So (i) $d = 3$, (ii) $a = -80$ and $b = 23$, that is, $-80 \times 213 + 23 \times 741 = 3$ and (iii) $m_1 = 71$ and $n_1 = 247$, that is, $213 = 71 \times 3$ and $741 = 247 \times 3$.

c) $m = 3146$, $n = 2783$.

$$\left.\begin{array}{cc|c} 1 & 0 & 3146 \\ 0 & 1 & 2783 \end{array}\right. \xrightarrow{R_1 - R_2} \left.\begin{array}{cc|c} 1 & -1 & 363 \\ 0 & 1 & 2783 \end{array}\right. \xrightarrow{R_2 - 7R_1} \left.\begin{array}{cc|c} 1 & -1 & 363 \\ -7 & 8 & 242 \end{array}\right. \xrightarrow{R_1 - R_2} \left.\begin{array}{cc|c} 8 & -9 & 121 \\ -7 & 8 & 242 \end{array}\right. \xrightarrow{R_3 - 2R_1} \left.\begin{array}{cc|c} 8 & -9 & 121 \\ -23 & 26 & 0 \end{array}\right.$$

So (i) $d = 121$, (ii) $a = 8$ and $b = -9$, that is, $8 \times 3146 - 9 \times 2783 = 121$ and (iii) $m_1 = 26$ and $n_1 = 23$, that is, $3146 = 26 \times 121$ and $2783 = 23 \times 121$.

In each case, the numbers n_1 and m_1 are given by the new row in the last matrix, in that order. Just omit any minus signs.

2. a) $m = 77$, $n = 21$.

$$\left.\begin{array}{cc|c} 1 & 0 & 77 \\ 0 & 1 & 21 \end{array}\right. \xrightarrow{R_1 - 3R_2} \left.\begin{array}{cc|c} 1 & -3 & 14 \\ 0 & 1 & 21 \end{array}\right. \xrightarrow{R_2 - R_1} \left.\begin{array}{cc|c} 1 & -3 & 14 \\ -1 & 4 & 7 \end{array}\right. \xrightarrow{R_1 - 2R_2} \left.\begin{array}{cc|c} 3 & -11 & 0 \\ -1 & 4 & 7 \end{array}\right.$$

So the *g.c.d.* is 7 and the l.c.m. is $3 \times 77 = 11 \times 21 = 231$.

Using factorisation into primes, $21 = 3 \times 7$ and $77 = 7 \times 11$. So the g.c.d is 7 and the l.c.m. is $3 \times 7 \times 11 = 231$.

b) $m = 357$, $n = 119$.

$$\left.\begin{array}{cc|c} 1 & 0 & 357 \\ 0 & 1 & 119 \end{array}\right. \xrightarrow{R_1 - 3R_2} \left.\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 119 \end{array}\right.$$

So the g.c.d. is 119 and the l.c.m is 357.

Using prime factorisation, $357 = 3 \times 119 = 3 \times 7 \times 17$ and $119 = 7 \times 17$. So the g.c.d. is 119 and the l.c.m. is 357, as before.

c) $m = 2898$, $n = 1495$.

$$\begin{array}{cc|c} 1 & 0 & 2898 \\ 0 & 1 & 1495 \end{array} \begin{array}{c} R_1 - R-2 \\ \rightarrow \end{array} \begin{array}{cc|c} 1 & -1 & 1403 \\ 0 & 1 & 1495 \end{array} \begin{array}{c} \rightarrow \\ R_2 - R_1 \end{array} \begin{array}{cc|c} 1 & -1 & 1403 \\ -1 & 2 & 92 \end{array} \begin{array}{c} R_1 - 15R_2 \\ \rightarrow \end{array} \begin{array}{cc|c} 16 & -31 & 23 \\ -1 & 2 & 92 \end{array}$$

$$\begin{array}{c} \rightarrow \\ R_2 - 4R_1 \end{array} \begin{array}{cc|c} 16 & -31 & 23 \\ -65 & 126 & 0 \end{array}$$

So the g.c.d. is 23 and the l.c.m. is $65 \times 2898 = 126 \times 1495 = 188370$

Using prime factorisation,

$$2898 = 2 \times 1449 = 2 \times 3^2 \times 161 = 2 \times 3^2 \times 7 \times 23$$

$$1495 = 5 \times 299 = 5 \times 13 \times 23$$

So the g.c.d. is 23 and the l.c.m. is $2 \times 3^2 \times 5 \times 7 \times 13 \times 23 = 10 \times 63 \times 13 \times 23 = 188370$.

3.

$$x^3 + 1 = (x+1)(x^2 - x + 1)$$

So if n is any integer $n+1$ and $n^2 - n + 1$ are divisors of $n^3 + 1$. So this gives

$$2^3 + 1 = 9 = (2+1)(4-2+1) = 3 \times 3, \quad 3^3 + 1 = 28 = (3+1)(3^2 - 3 + 1) = 4 \times 7 = 2^2 \times 7,$$

$$4^3+1 = 65 = (4+1)(4^2-4+1) = 5 \times 13, \quad 5^3+1 = 126 = (5+1)(5^2-5+1) = 6 \times 21 = 2 \times 3^2 \times 7,$$

$$6^3+1 = 217 = (6+1)(6^2-6+1) = 7 \times 31, \quad 7^3+1 = 344 = (7+1)(7^2-7+1) = 8 \times 43 = 2^3 \times 43,$$

$$8^3+1 = 513 = (8+1)(8^2-8+1) = 9 \times 57 = 3^3 \times 19, \quad 9^3+1 = 730 = (9+1)(9^2-9+1) = 10 \times 71 = 2 \times 5 \times 71,$$

$$10^3 + 1 = 1001 = (10+1)(10^2 - 10 + 1) = 11 \times 91 = 11 \times 13 \times 7.$$

The numbers $n+1$ and $n^2 - n + 1$ are divisors of $n^3 + 1$ for any $n \in \mathbb{Z}$, but not necessarily prime. So in some cases, further factorisation of these numbers was needed.

4. a) We have

$$(x-1) \cdot \left(\sum_{i=0}^{m-1} x^i \right) = \sum_{i=0}^{m-1} (x^{i+1} - x^i) = x - 1 + (x^2 - x) + \cdots (x^m - x^{m-1}) = x^m - 1$$

This is sometimes called the method of telescoping sums. *A formal induction proof was given in lectures a couple of weeks ago, which was not required, although it was not wrong to include it. This method can also be found in the notes.*

Putting $x = 10$ we obtain

$$9 \cdots \left(\sum_{i=0}^{m-1} 10^i \right) = 10^m - 1$$

for all $m \geq 1$, and so $9 | 10^m - 1$ for all $m \geq 1$. Since $10^0 - 1 = 0$ we also have $9 | 10^0 - 1$.

Applying the test to 216, $2 + 1 + 6 = 9$ is divisible by 9 and $216 = 9 \times 24$.

Applying the test to 361125, $3+6+1+1+2+5 = 18$ which is 2×9, and $361125 = 9 \times 40125$.

b) As in the previous question, we have

$$(x+1)(x^2 - x + 1) = x^3 + x^2 - x^2 - x + x + 1 = x^3 + 1$$

Applying this with $x = 10$ gives

$$1001 = 10^3 + 1 = (10+1) \cdot (10^2 - 10 + 1) = 11 \times 91 = 11 \times 7 \times 13$$

So $13|10^3+1$. Also

$$10^6-1=(10^3-1)(10^3+1)$$

and $13|10^6-1=999999$.

Applying the test to $1495=115\times 13$

$$5-1-3\times 9-4\times 4=4-27-16=-39=-3\times 13$$

Applying the test to $74802=5754\times 13$,

$$2-4-3\cdot(0-7)-4\times 8=-2+21-32=-13$$

a_1 is the last digit of the number, a_2 next to the last, and so on. So for 1495, for example, we have $a_1=5$, $a_2=9$, $a_3=4$, $a_4=1$. As it happens, by coincidence, $5941=467\times 13$ is also divisible by 13 – which caused some confusion, because it was possible to get the digits in the wrong order and thus apply the test incorrectly to 1495 and still get confirmation that it is divisible by 13.

Solutions to Practice Problems

5. a) $m=168$, $n=63$.

$$\begin{array}{cc|c} 1 & 0 & 168 \\ 0 & 1 & 63 \end{array} \quad \begin{array}{c} R_1-2R_1 \\ \to \end{array} \quad \begin{array}{cc|c} 1 & -2 & 42 \\ 0 & 1 & 63 \end{array} \quad \begin{array}{c} \to \\ R_2-R_1 \end{array} \quad \begin{array}{cc|c} 1 & -2 & 42 \\ -1 & 3 & 21 \end{array} \quad \begin{array}{c} R_1-2R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 3 & -8 & 0 \\ -1 & 3 & 21 \end{array}$$

So (i) $d=21$, (ii) $a=-1$ and $b=3$, that is, $-1\times 168+3\times 63=7$ and (iii) $m_1=8$ and $n_1=3$, that is, $168=8\times 21$ and $63=3\times 21$.

Using prime factorisation, $168=8\times 21=2^3\times 3\times 7$, while $63=3^2\times 7$. So the gcd is $3\times 7=21$ and the lcm is $2^3\times 3^2\times 7=504$.

b) $m=234$, $n=416$.

$$\begin{array}{cc|c} 1 & 0 & 234 \\ 0 & 1 & 416 \end{array} \quad \begin{array}{c} \to \\ R_2-R_1 \end{array} \quad \begin{array}{cc|c} 1 & 0 & 234 \\ -1 & 1 & 182 \end{array} \quad \begin{array}{c} R_1-R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 2 & -1 & 52 \\ -1 & 1 & 182 \end{array} \quad \begin{array}{c} \to \\ R_2-3R_1 \end{array} \quad \begin{array}{cc|c} 2 & -1 & 52 \\ -7 & 4 & 26 \end{array}$$

$$\begin{array}{c} R_1-2R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 16 & -9 & 0 \\ -7 & 4 & 26 \end{array}$$

So (i) $d=26$, (ii) $a=-7$ and $b=4$, that is, $-7\times 234+4\times 416=26$ and (iii) $m_1=9$ and $n_1=16$, that is, $234=9\times 26$ and $416=16\times 26$.

Using prime factorisation, $234=2\times 117=2\times 9\times 13=2\times 3^2\times 13$, and $416=4\times 104=4\times 8\times 13=2^5\times 13$. So the gcd is $2\times 13=26$ and the lcm is $2^5\times 3^2\times 13=3774$.

c) $m=543$, $n=1251$.

$$\begin{array}{cc|c} 1 & 0 & 543 \\ 0 & 1 & 1251 \end{array} \quad \begin{array}{c} \to \\ R_2-2R_1 \end{array} \quad \begin{array}{cc|c} 1 & 0 & 543 \\ -2 & 1 & 165 \end{array} \quad \begin{array}{c} R_1-3R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 7 & -3 & 48 \\ -2 & 1 & 165 \end{array} \quad \begin{array}{c} \to \\ R_2-3R_1 \end{array} \quad \begin{array}{cc|c} 7 & -3 & 48 \\ -23 & 10 & 21 \end{array}$$

$$\begin{array}{c} R_1-2R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 53 & -23 & 6 \\ -23 & 10 & 21 \end{array} \quad \begin{array}{c} R_1-3R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 53 & -23 & 6 \\ -182 & 79 & 3 \end{array} \quad \begin{array}{c} R_1-2R_2 \\ \to \end{array} \quad \begin{array}{cc|c} 417 & -181 & 0 \\ -182 & 79 & 3 \end{array}$$

So (i) $d=3$, (ii) $a=-182$ and $b=79$, that is, $-182\times 543+79\times 1251=4$ and (iii) $m_1=181$ and $n_1=417$, that is, $543=181\times 3$ and $1251=417\times 3$.

Using prime factorisation, $543=3\times 181$, and 181 is prime. We can check this from the list of the first 1000 primes, or, alternatively, by checking that it is not divisible by 2, 3, 5, 7, 11 and 13.

Also, $1251 = 9 \times 139 = 3^2 \times 139$, and 139 is prime. So 3 is the gcd, and $3^2 \times 139 \times 181 = 226431$ is the lcm.

6. $x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$. Putting $x = 10$, we have

$$10^4 - 1 = (10^2 + 1) \times (10 - 1) \times (10 + 1) = 101 \times 9 \times 11 = 101 \times 3^2 \times 11.$$

The last expression in the line above is the prime factorisation of $10^4 - 1 = 9999$.

7. We start by checking Binet's formula for $n = 1$ and $n = 2$:

$$u_1 = 1 = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1,$$

$$u_2 = \frac{((1+\sqrt{5})^2 - (1-\sqrt{5})^2}{4\sqrt{5}} = \frac{1+5+2\sqrt{5}-1-5+2\sqrt{5}}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

Suppose the formula works for u_k for $1 \le k \le n$ where $n \ge 2$. Then

$$u_{n+1} = u_{n-1} + u_n = \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^{n-1}(2+1+\sqrt{5})}{2^n\sqrt{5}} - \frac{(1-\sqrt{5})^{n-1}(2+1-\sqrt{5}}{2^n\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^{n-1}(6+2\sqrt{5}) - (1-\sqrt{5})^{n-1}(6-2\sqrt{5})}{2^{n+1}\sqrt{5}} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

because $6 + 2\sqrt{5} = (1+\sqrt{5})^2$ and $6 - 2\sqrt{5} = (1-\sqrt{5})^2$

So if the formula holds for $1 \le k \le n$, then it holds for $1 \le k \le n+1$. So by induction, Binet's formula holds for all $n \in \mathbb{N}$.

Now we compute u_n and $(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$ for $n \le 12$.

n	u_n	$(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$	n	u_n	$(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$
1	1	0.72360679775	7	13	12.98459713475
2	1	1.1708239325	8	21	21.0095149426
3	2	1.894427191	9	34	33.99411662902
4	3	3.06524758425	10	55	55.00363612328
5	5	4.95967477525	11	89	88.99775275231
6	8	8.0249223595	12	144	144.00138887561

The two quantities u_n and $(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$ are getting closer. This is because $(1-\sqrt{5})^n 2^{-n}/\sqrt{5}$ is getting closer to 0. in fact for $n = 12$ this is 0.001388875..