MATH105 Problem Sheet 9: Real numbers
Due Wednesday 3rd December

1. Mimic the proof of irrationality of $\sqrt{2}$ to show that $4^{1/3}$ is not rational, where this is defined to be the unique number $x > 0$ such that $x^3 = 4$.

2. Improve on the Indian altar approximation of $\frac{577}{408}$ to $\sqrt{2}$ by computing $(\frac{577}{408})^2 - 2$ or otherwise.

3. An element a of a set A is *maximal* if $x \leq a$ for all $x \in A$. Similarly, $a \in A$ is *minimal* if $a \leq x$ for all $x \in A$. If a maximal element exists it is unique, and similarly for a minimal element. Determine which of the following sets A has a maximal element and which has a minimal element.

a) $A = \{\frac{1}{n^2} : n \in \mathbb{Z}_+\}$

b) $A = \{x \in \mathbb{R} : x^2 \leq 3\}$

c) $A = \{x \in \mathbb{R} : x^2 < 5\}$

d) $A = \{x \in \mathbb{Q} : x^2 \leq 5\}$

e) $A = \{x \in \mathbb{R} : 1 - x^2 \leq 2\}$.

4. Show that if $x^2 < 3$ and $1 \leq x$ and $0 < \varepsilon < 1$ then $(x + \varepsilon)^2 < x^2 + 3x\varepsilon$. Hence show that if, in addition $\varepsilon < (3 - x^2)/(3x)$ then $(x + \varepsilon)^2 < 3$.

Determine which of the following sets A are Dedekind cuts. You may assume that $\sqrt{3}$ is irrational. You may want to use the above to show that A has no maximal element, in some of the parts.

a) $A = \{x \in \mathbb{Q} : x < 2\}$

b) $A = \{x \in \mathbb{Q} : -x \geq 1\}$

c) $A = \{x \in Q : x^2 < 3 \vee x < 1\}$

d) $A = \{x \in \mathbb{Q} : x^2 < 3 \vee x > 1\}$

e) $A = \{x \in \mathbb{Q} : x^2 < 3\}$

I will collect solutions at the lecture on Wednesday 3rd December. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.

Possible tutorial problems

Some or all of these may be used in the problem class, if desired. They can also used as practice problems, either for practising for the homework problems, or for later practice during revision, or both.

5. Mimic the proof of irrationality of $\sqrt{p}$ to show that $5^{1/3}$ is not rational, where this is defined to be the unique number $x > 0$ such that $x^3 = 5$.

6. The method of approximation in the Indian altar problem can be adapted to find $\sqrt{p}$ for any integer p. Suppose we want to approximate $\sqrt{3}$. It is reasonable to take $x_0 = 2$ because $2^2 > 4$ and $1^2 = 1 < 3$. Now define

$$x_1 = x_0 - \frac{(x_0^2 - 3)}{2x_0} = 1 + \frac{3}{4}.$$

Then

$$x_1^2 = \frac{7^2}{4^2} = \frac{49}{16} = 3 + \frac{1}{16}.$$

Compute x_2, x_3 and $x_3^2 - 3$, where

$$x_{n+1} = x_n - \frac{(x_n^2 - 3)}{2x_n}.$$

Some people may have already have noticed that this approximation method is an example of what we call the *Newton-Raphson method.*

7. An element a of a set A is *maximal* if $x \leq a$ for all $x \in A$. Similarly, $a \in A$ is *minimal* if $a \leq x$ for all $x \in A$. If a maximal element exists it is unique, and similarly for a minimal element. Determine which of the following sets A has a maximal element and which has a minimal element.

a) $A = \{-\frac{1}{n^3} : n \in \mathbb{Z}_+\}$

b) $A = \{x \in \mathbb{R} : x^2 \leq 5\}$

c) $A = \{x \in \mathbb{R} : x^2 < 3\}$

d) $A = \{x \in \mathbb{Q} : x^2 \leq 3\}$

e) $A = \{x \in \mathbb{R} : 2 - x^2 \leq 3\}$.

8. Show that if $x^2 < 5$ and $1 \leq x$ and $0 < \varepsilon < 1$ then $(x + \varepsilon)^2 < x^2 + 3x\varepsilon$. Hence show that if, in addition $\varepsilon < (5 - x^2)/(3x)$ then $(x + \varepsilon)^2 < 5$.

Determine which of the following sets A are Dedekind cuts. You may assume that $\sqrt{5}$ is irrational. You may want to use the above to show that A has no maximal element, in some of the parts.

a) $A = \{x \in \mathbb{Q} : x < 3\}$

b) $A = \{x \in \mathbb{Q} : -x \geq 2\}$

c) $A = \{x \in Q : x^2 < 5 \vee x < 1\}$

d) $A = \{x \in \mathbb{Q} : x^2 < 5 \vee x > 1\}$

e) $A = \{x \in \mathbb{Q} : x^2 < 5\}$

9. Remember that a nonempty finite set is of the form $\{f(m) : m \in \mathbb{Z}_+, m \leq n\}$ for some injective function f and some $n \in \mathbb{Z}_+$. Prove by induction on n that every nonempty finite set of real numbers has a maximal element and a minimal element.

Hint: It may be a little formal to write a finite set in the form $\{f(m) : m \in \mathbb{Z}_+, m \leq n\}$ where $f : \{m \in Z_+ : m \leq n\} \to \mathbb{R}$ is a function, but it probably helps to write down the induction. The base case in this question is $n = 1$. If $f : \{m \in Z_+ : m \leq n\} \to \mathbb{R}$ is a function and $n > 1$ then $g : \{m \in Z_+ : m \leq n - 1\} \to \mathbb{R}$ is also a function, where $g(m) = f(m)$ for $m \leq n - 1$.